Recall: \( P_k = \lim_{t \to \infty} P_k(t) \) (well defined for irreducible, pos. recurrent CTMC)

Today: Calculate \( \{P_k\} \) for birth-death processes

Q: What are \( P_k \) for rate \( \lambda \) Poisson process \( \Lambda \)?
A: \( \Lambda(t) \to \infty \) as \( t \to \infty \). Thus \( \lim_{t \to \infty} P_k(t) = 0 \).
\( \Rightarrow \) \( \sum_{k=0}^\infty P_k = 0 \).

Why? Null recurrent (and not irreducible).

Q: What are \( P_k \) for birth-death process w/ non-zero death rates \( \mu_1, \mu_2, \ldots \) \( (\mu_0 = 0) \)

Recall:
\[ V_j = \lambda_j + \mu_j \]
Rate leave \( j \) = Rate enter \( j \)
\[ V_j P_j = \sum_{k \neq j} q_{kj} P_k \]
\[ q_{kj} = \begin{cases} \lambda_k & j = k+1 \\ \mu_k & j = k-1 \\ 0 & \text{else} \end{cases} \]
\[ = q_{j-1,j} P_{j-1} + q_{j+1,j} P_{j+1} \]
\[ = \lambda_{j-1} P_{j-1} + \mu_{j+1} P_{j+1} \]

\[ j = 0 : \quad \lambda_0 P_0 = \mu_1 P_1 \]
\[ j = 1 : \quad (\lambda_1 + \mu_1) P_1 = \lambda_0 P_0 + \mu_2 P_2 \]
\[ j = 2 : \quad (\lambda_2 + \mu_2) P_2 = \lambda_1 P_1 + \mu_3 P_3 \]
\[ \vdots \]
\[ j = n : \quad (\lambda_n + \mu_n) P_n = \lambda_{n-1} P_{n-1} + \mu_{n+1} P_{n+1} \]
Solve: \( j = 0 \) : \( P_0 = \frac{\lambda_0}{\mu_1} P_0 \)

Plug \( j = 0 \) into \( j = 1 \) \( \Rightarrow \) \( \lambda_1 P_1 = \mu_2 P_2 \)

\( \Rightarrow P_2 = \frac{\lambda_1}{\mu_2}. P_1 = \frac{\lambda_1}{\mu_2}. \frac{\lambda_0}{\mu_1}. P_0 \)

Similarly, \( P_3 = \frac{\lambda_2}{\mu_3}. \frac{\lambda_1}{\mu_2}. \frac{\lambda_0}{\mu_1}. P_0 \)

and \( P_n = \Gamma_n P_0 \) where \( \Gamma_n = \frac{\lambda_{n-1}}{\mu_n} \cdot \frac{\lambda_{n-2}}{\mu_{n-1}} \cdots \frac{\lambda_0}{\mu_1} \)

Now, use \( \sum_{n=0}^{\infty} P_n = P_0 + P_1 \sum_{n=1}^{\infty} \Gamma_n \)

\( \Rightarrow P_0 = \frac{\Gamma_0}{1 + \sum_{n=1}^{\infty} \Gamma_n} \)

\( P_n = \Gamma_n P_0 = \frac{\Gamma_n}{1 + \sum_{n=1}^{\infty} \Gamma_n} \)

Example:

Join line

at rate \( \lambda \)

Consider a single server who receives customers at a rate \( \lambda \) Poisson process.

Service times are \( \text{Exp}(\mu) \)

\( \{X(t) : t \geq 0\} = \# \) of customers in queue at time \( t \)

(in service & waiting)

This is called an \( M/M/1 \) queue.

It is a birth-death process w/ \( \lambda_n = \lambda, n \geq 0, \mu_n = \mu, n \geq 1 \) \( (\mu_0 = 0) \).

In machinery above, \( \Gamma_n = \frac{\lambda^n}{\mu^n} \) so
Case 1: \( m > \lambda \) so \( \frac{\lambda}{m} < 1 \)

Then, \( \sum_{n=1}^{\infty} r_n = \sum_{n=1}^{\infty} \left( \frac{\lambda}{m} \right)^n = \frac{\lambda}{m - \lambda} = \frac{1}{1 - \frac{\lambda}{m}} \)

Thus, \( P_0 = 1 - \frac{\lambda}{m} \) & \( P_n = \left( \frac{\lambda}{m} \right)^n \left( 1 - \frac{\lambda}{m} \right) \)

Case 2: \( m \geq \lambda \), \( \sum_{n=1}^{\infty} r_n = \infty \Rightarrow P_n = 0 \ \forall n. \)

Q: Why?
A: \( \lambda > m \)

\( \lambda = m \) Null recurrent, just like symmetric 1-d random walk.