Limiting probabilities for CTMC

Suppose $P_j = \lim_{t \to \infty} P_{ij}(t)$ exists and does not depend on $i$.

Q: $P_j = ?$ (Find eqns $\{P_j\}$ must satisfy.)

First attempt: Use backward Kol. eqn:

$$P_{ij}(t) = \sum_{k+i} q_{ik} P_{kj}(t) - \nu_i P_{ij}(t)$$

$$\lim_{t \to 0} P_{ij}(t) \to P_j(t) \quad \text{(by assumption)}$$
$$\lim_{t \to 0} P_{ij}(t) \to 0 \quad \text{(follows from (1))}$$

Thus, backward Kol. eqn gives:

$$0 = \sum_{k+i} q_{ik} P_j - \nu_i P_j$$

$$= \nu_i P_j - \nu_i P_j = 0$$

Not helpful!

Attempt 2:

Forward Kolmogorov eqn:

$$P_{ij}(t) = \sum_{k+j} q_{ik} P_{kj}(t) - \nu_i P_{ij}(t)$$

Prove (tweak pf of backward Kol. eqn)

Now take limit as $t \to 0$: 
\[ 0 = \sum_{k \neq j} P_k \cdot r_{kj} - V_j \cdot P_j \]

**Eqs for \( \{P_j\}^2 \):**

\[ V_j \cdot P_j = \sum_{k \neq j} q_{kj} \cdot P_k, \quad \sum_j P_j = 1 \quad (\star) \]

**Ex:** Two state MC.

\[
\begin{align*}
V_1 &= \lambda, & q_{10} &= \mu \\
V_0 &= \lambda, & q_{01} &= \lambda \\
\end{align*}
\]

\( \text{Ex:} \quad 1 \Rightarrow \text{Exp}(\mu) \)

**(\star) becomes:**

\[
V_0 \cdot P_0 = q_{10} \cdot P_1
\]

\[
\lambda P_0 = \mu P_1
\]

\[
P_0 + P_1 = 1
\]

\[
P_i = \frac{1}{\text{Exp}(A)} = \frac{\lambda}{\lambda + \mu}
\]

\[
P_0 = 1 - P_1 = \frac{\mu}{\lambda + \mu}
\]

**Theorem:** For a positive recurrent, irreducible MC

1. The limiting probabilities \( \{P_j\} \) exist and obey (\star).
2. \( P_j \) = long run proportion of time chain is in state \( j \).

**Interpretations of (\star):**

**A)** Analog to eqa for stationary dist of discrete time MC.

\[
\text{Recall } \pi_T = \pi_P & \quad \sum_i \pi_i = 1 \\
\Rightarrow \pi_j = \frac{\pi_j \cdot P_j}{P_k} \]

**B)** LHS = \( V_j \cdot P_k \) = rate leaving \( k \)

RHS = \( \sum_{k \neq j} q_{kj} \cdot P_j \) = rate coming into \( k \)

**must cancel for stationary behavior**