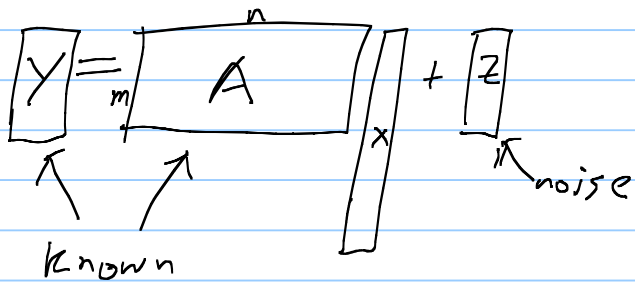
Compressed sensing w/ noise

Note Title 2015-11-30

4 $\boxed{\text{Precise analysis of compressed sensing}}$

Model:



- $x \in K_s := \{ x \in \mathbb{R}^n : \|x\|_0 \le s \le \frac{n}{8} \}$

Goal: Estimate x.

Q: Does any method work better than ℓ_1 minimization?

A: No*.

* Some caveat tbd.

Today: Analyze ℓ_1 minimization.

Let $\hat{x}$ be sol$^{\underline{n}}$ to:

$$\min \|Ax' - y\|_2^2 \quad \text{s.t.} \quad \|x'\|_1 \le \|x\|_1$$

Thm Let A have ind. sub-Gauss. entries satisfying $\mathbb{E}A_{ij} = 0$, $\mathbb{E}A_{ij}^2 = 1$, $\|A_{ij}\|_{\psi_2} \le c$. Let* $\|z\|_2 \le c\sqrt{m}\,\sigma$. Suppose $m \ge C s \log(\frac{n}{s})$. Then, w.p. $\ge .98$,

$$\|\hat{x} - x\|_2^2 \le \frac{C s \log(\frac{n}{s}) \sigma^2}{m}$$

Remarks: ① We assume $\|z\|_2^2 \le C m \sigma^2$, which, roughly allows $|z_i| \sim \sigma$. z does not have to be random, but if it is, it must be indep. of A.

② $s\log(\frac{n}{s})$ is dimension of $K_s \cap B_2^n$ as measured by metric entropy: $\log(N(K_s \cap B_2^n), \frac{1}{2}B_2^n)$.

Thus, result states informally "If # of meas. exceeds dimension then error is proportional to

$$\frac{\text{dimension}\cdot(\text{noise level})}{\#\text{ of measurements}}."$$

Proof of thm:

Let $h = \hat{x} - x$.

$$\boxed{\begin{array}{l}\textbf{Lemma}\quad h \text{ satisfies the following two constraints:}\\ ①\ \|h\|_1 \leq 2\sqrt{s}\,\|h\|_2 \\ ②\ \|Ah\|_2^2 \leq 2\langle Ah, z\rangle\end{array}}$$

Proof ① Let $T \subseteq \{1, \ldots, n\}$ be the support of x, T^c be the complement. For vector v, let $v_T \in \mathbb{R}^{|T|}$ be v restricted to T.

Note: a) $\|\hat{x}\|_1 \leq \|x\|_1 = \|x_T\|_1$ (by def)

Also b) $\|\hat{x}\|_1 = \|h + x\|_1 = \|h_T + x_T\|_1 + \|h_{T^c}\|_1$.

$$\geq \|x_T\|_1 - \|h_T\|_1 + \|h_{T^c}\|_1$$

Combine a) & b) to give: $\|h_{T^c}\|_1 \leq \|h_T\|_1$

$$\Rightarrow \|h\|_1 = \|h_{T^c}\|_1 + \|h_T\|_1 \leq 2\|h_T\|_1$$

$$\leq 2\sqrt{s}\,\|h_T\|_2 \quad \text{(By Cauchy-Schwartz ≠)}$$

$$\leq 2\sqrt{s}\,\|h\|_2 \quad ■$$

(2) $\|A\hat{x}-y\|_2^2 \le \|Ax-y\|_2^2 = \cancel{\|z\|_2^2}$ by def:

$\|A\hat{x}-Ax-z\|_2^2 = \|Ah-z\|_2^2 = \langle Ah-z, Ah-z\rangle$

$= \|Ah\|_2^2 - 2\langle Ah, z\rangle + \cancel{\|z\|_2^2}$ ■

Now we use lemma to prove thm:

Let $D = \{v \in \mathbb{R}^n : \|v\|_1 \le 2\sqrt{s}\|v\|_2\}$.

Consider LHS of (2). By master theorem, A is well conditioned when restricted to D provided $m \ge C\, w(D \cap B_2^n)^2$ i.e. w.p. $\ge .99$

(A) $\|Ah\|_2^2 \ge \frac{m}{2}\|h\|_2^2$

Now, consider the RHS of (2).

$\langle Ah, z\rangle = \langle h, A^T z\rangle = \|h\|_2 \cdot \|z\|_2 \cdot \langle \bar{h}, A^T \bar{z}\rangle$ (*)

where $\bar{h} = \frac{h}{\|h\|_2}$, $\bar{z} = \frac{z}{\|z\|_2}$

Note: $A^T\bar{z}$ is a sub-Gauss vector &

$\mathbb{E}\langle \bar{h}, A^T\bar{z}\rangle \le \mathbb{E} \sup_{v \in D \cap B_2^n} \langle v, A^T \bar{z}\rangle \le C\, w(D \cap B_2^n)$

↑ Gauss. sub-Gauss. comparison

Plug into (*), & combine w/ Markov ≠ to give

(B) $\langle Ah, z\rangle \le C \cdot \|h\|_2 \cdot \|z\|_2 \cdot w(D \cap B_2^n)$ w.p. $\ge .99$

Combine (A) & (B) w/ (2) to give:

$\frac{m}{2}\|h\|_2^{\cancel{2}} \le \|Ah\|_2^2 \le C\cancel{\|h\|_2}\|z\|_2\, w(D \cap B_2^n)$ w.p. $> .98$

The theorem is completed by controlling the mean width:

Lemma $\boxed{w(D \cap B_2^n) \le C\sqrt{s \log\left(\frac{n}{s}\right)}}$

Proof: Note:

For $h \in D \cap B_2^n$, $\|h\|_1 \le 2\sqrt{s}\,\|h\|_2 \le 2\sqrt{s}$

Thus $D \cap B_2^n \subseteq (2\sqrt{s}\,B_1^n) \cap B_2^n \underset{\uparrow}{\subseteq} 2\,\mathrm{conv}(K_{4s} \cap B_2^n)$

Exercize

$$\Rightarrow w(D \cap B_2^n) \le 2w\left(\mathrm{conv}(K_{4s} \cap B_2^n)\right)$$

$$= 2w(K_{4s} \cap B_2^n)$$ (mean width of convex hull is same as that of original set)

(by hw) $$= C s \log\left(\tfrac{n}{s}\right)$$

∎