Poisson processes, using what we've learned.

Q: Let $\xi_N(t):t \geq 0$ be a rate $\lambda$ Poisson process. Let $S_1, S_2, \ldots$ be time of $i$th event. Find

$$P(S_1, S_2, S_3 \in [0, 1] \mid N(10) = 3),$$

A: Given $N(3) = 3$, first 3 events occur at uniform at random times in $[0, 10]$. Let $U_1, U_2, U_3 \sim \text{Unif}[0, 3]$. We calculate

$$P(U_1, U_2, U_3 \in [0, 1]) = P(U_i \in [0, 1])^3 = \left(\frac{1}{10}\right)^3 = \frac{1}{1000}.$$ 

Application (HIV analytics)

Model

- individuals contract HIV as a rate $\lambda$ Poisson process
- time from infection to onset of symptoms is r.v. w/ known cdf $G$.

$\hat{N}_1(t) = \#$ who have had onset of symptoms

$\hat{N}_2(t) = \#$ who are HIV positive w/o symptoms

Goal: Estimate $\mathbb{E} N_2(t)$.

Step: Consider fixed $t > 0$ and consider fractured Poisson process

Type 1: Infected individual shows symptoms by time $t$.

Type 2: II $\Rightarrow$ doesn't show II $\Rightarrow$ II.

Individual infected at time set is Type 1 or
\[ P_i(s) = P(\text{onset of symptoms within } t-s \text{ of infection}) = G(t-s) \]

Type 2 w/ prob \( 1 - P_i(s) = 1 - G(t-s) = \tilde{G}(t-s) \)

\[ \Rightarrow N_2(t) \sim \text{Poisson} \left( \lambda \int_0^t P_i(s) \, ds \right) \]

\[ \text{Note} \quad \int_0^t P_i(s) \, ds = \int_0^t G(t-s) \, ds = \int_0^t \tilde{G}(u) \, du \]

\[ \Rightarrow N_2(t) \sim \text{Poisson} \left( \lambda \int_0^t \tilde{G}(u) \, du \right) \]

Similarly, \( N_1(t) \sim \text{Poisson} \left( \lambda \int_0^t \tilde{G}(u) \, du \right) \)

Need to estimate \( \lambda : N_1(t) = n_1 \) is observed.

Estimate: \( n_1 \approx E N_1(t) = \lambda \int_0^t \tilde{G}(u) \, du \)

Estimate of \( \lambda : \hat{\lambda} = \frac{n_1}{\lambda \int_0^t \tilde{G}(u) \, du} \)

Estimate of \( N_2(t) : \hat{N}_2 = \hat{\lambda} \int_0^t \tilde{G}(u) \, du \]

\[ = \frac{n_1 \int_0^t \tilde{G}(u) \, du}{\int_0^t \tilde{G}(u) \, du} \]

E.g. \( t = 16 \text{ yrs}, \quad G \sim \text{Exp} \left( \frac{1}{16 \text{ yrs}} \right) \quad n_1 = 220,000 \)

\[ \hat{N}_2 = 220,000 \frac{\int_0^{16} \, e^{-u} \, du}{\int_0^{16} \left( 1 - e^{-u} \right) \, du} \approx 718,959 \]