Dvoretzky-Milman theorem

3 Some key ideas from geometric functional analysis.

<u>DMT</u>: "Any norm $\|\cdot\|$ on $\mathbb{R}^n$ is almost Euclidean on a random subspace E."
I.e.

$$\|x\|_2 \approx \underset{\substack{\uparrow \\ \text{proportionality} \\ \text{coefficient}}}{M} \|x\|_2 \qquad \forall x \in E$$

<u>Value of M?</u> $\quad \|x\| \approx M \quad \forall\, x \in S^{n-1} \cap E$

$$M := \underset{\substack{\uparrow \\ \sim \mathrm{Unif}(S^{n-1})}}{\mathbb{E}\|x\|} = \int_{S^{n-1}} \|x\|\, d\sigma_n(x)$$

σ_n: normalized Lebesgue measure

<u>Dvoretzky-Milman Thm '73</u>

Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ s.t. $\|x\| \le b\|x\|_2 \ \forall x \in \mathbb{R}^n$.
Set

$$M := \mathbb{E}\|x\|, \quad x \sim \mathrm{Unif}(S^{n-1}).$$

Then, $\forall \epsilon > 0$, a random subspace $E \in G_{n,k}$ (grassmannian) of dimension $k = c_\epsilon (M/b)^2 n$ satisfies w.p. $\geq 1 - 2e^{-k}$ that

$$(1-\epsilon) M\|x\|_2 \le \|x\| \le (1+\epsilon) M\|x\|_2 \qquad \forall x \in E.$$

<u>Remark</u> $\quad b := \max_{x \in S^{n-1}} \|x\|. \qquad M = \operatorname{avg}_{x \in S^{n-1}} \|x\|$

$\frac{M}{b} < 1$ measures how far $\|\cdot\|$ is from Euclidean.

Ex) If $\frac{1}{2}\|x\|_2 \le \|x\| \le 2\|x\|_2 \Rightarrow \left(\frac{M}{b}\right) \geq \frac{1}{4}$

$k \sim n$

Ex) $\|\cdot\| = \|\cdot\|_1$

$$\|\cdot\|_1 \le \underbrace{\sqrt{n}}_{= b} \|\cdot\|_2$$

Calculate M:

$$\mathbb{E}_{\substack{x \sim \text{unif}(S^{n-1})}} \|x\|_1 = ?$$

Replace w/ Gaussian expectation. Let $g \sim N(0, I_n)$

$$g = \underset{\text{unif}(S^{n-1})}{x} \cdot \|g\|_2 \quad \text{indep of } x \qquad x = \frac{g}{\|g\|_2}$$

$$\mathbb{E}\|g\|_1 = \mathbb{E}\|x\|g\|_2\|_1 = \mathbb{E}\|x\|_1 \cdot \|g\|_2 \overset{\text{indep}}{=} \mathbb{E}\|x\|_1 \cdot \mathbb{E}\|g\|_2$$

$$= \mathbb{E}\|x\|_1 \cdot (\sqrt{n} + o(1))$$

$$\Rightarrow \mathbb{E}\|x\|_1 = \frac{\mathbb{E}\|g\|_1}{(\sqrt{n}+o(1))} = \frac{\sqrt{\frac{2}{\pi}} \cdot n}{\sqrt{n}+o(1)} \approx \boxed{\sqrt{\frac{2}{\pi}} \cdot \sqrt{n} \approx M}$$

$$\Rightarrow \frac{M}{b} \geq c.$$

Thus, $\|\cdot\|_1$ is almost Euclidean on a $c \cdot n$ dimensional subspace!

Ex) $\|\cdot\| = \|\cdot\|_\infty \qquad b = 1$

$$\mathbb{E}\|x\|_\infty \approx \frac{\mathbb{E}\|g\|_\infty}{\sqrt{n}} \approx c\frac{\sqrt{\log n}}{\sqrt{n}} = M$$

$$\left(\frac{M}{b}\right)^2 \geq \frac{\log n}{n}$$

$\|\cdot\|_\infty$ is almost Euclidean on a $c \log n$ dimensional subspace ☹.

Assume ✓

Proof of DMT: (simple covering argument)

① Cover
② Deviation $\neq$
③ Union bound
④ Continuation argument

Sample E as follows: Fix $\mathbb{R}^k = \text{span}\{e_1, \dots, e_k\} \subseteq \mathbb{R}^n$
Let $U \in O(n)$ be random uniform. Take
$E = U(\mathbb{R}^k)$.

① Cover: Let $\mathcal{N}_1$ be an ϵ-cover of (Using Euclidean balls)
$\mathbb{R}^k \cap S^{n-1}$ w/ $|\mathcal{N}_1| \le \left(\frac{3}{\epsilon}\right)^k$. Then

$\mathcal{N}_2 := U(\mathcal{N}_1)$ is an ϵ-cover of $E \cap S^{n-1}$
w/ $|\mathcal{N}_2| \le \left(\frac{3}{\epsilon}\right)^k$.

② Deviation $\neq$: Fix $x \sim \mathcal{N}_1$. Let $y = Ux \in \mathcal{N}_2$.
Note $y \sim \text{Unif}(S^{n-1})$.

Let $f(z) := \|z\|$.

By assumption f is b-Lipschitz.
Assume $b = 1$ for simplicity.

Then, by spherical concentration

$$\|f(y) - \mathbb{E} f(y)\|_{\psi_2} = \|\|y\| - M\|_{\psi_2} \le \frac{c}{\sqrt{n}}$$

i.e. $P(|\|y\| - M| > t) \le 2\exp(-ct^2 n)$

$$\Leftrightarrow P(|\|y\| - M| > M\epsilon) \le 2\exp(-c M^2 \epsilon^2 n)$$

③ Union bound:

$$P\left(\max_{y \in \mathcal{N}_2} |\|y\| - M| > M\epsilon\right) \le 2\left(\frac{3}{\epsilon}\right)^k \exp(-c M^2 \epsilon^2 n)$$

$$= 2\exp\left(k \log\left(\tfrac{3}{\epsilon}\right) - c M^2 \epsilon^2 n\right)$$

(By def of k) $\quad \le 2\exp(-k)$

$\Rightarrow$ w.p. $\geq 1 - 2\exp(-k)$,

$M(1-\epsilon) \leq \|y\| \leq M(1+\epsilon)$ on the net $\mathcal{N}_2$.

(4) Continuation argument.
Let $z \in S^{n-1} \cap E$. Then $\exists$ $\bar{z} \in \mathcal{N}_2$ s.t.

$\|\bar{z} - z\|_2 \leq \epsilon$ Set $w = \frac{\bar{z} - z}{\|\bar{z} - z\|_2}$

$\Rightarrow \|z\| \leq \|\bar{z}\| + \|z - \bar{z}\| \leq \|\bar{z}\| + \epsilon\|w\|$ $(*)$

Set $a := \sup_{z \in S^{n-1} \cap E} \|z\|$

Take supremums on both sides of $(*)$:

$$a \leq \max_{\bar{z} \in \mathcal{N}_2} \|\bar{z}\| + \epsilon a$$

$\Rightarrow (1-\epsilon) a \leq M(1+\epsilon)$ w.h.p.
$\Rightarrow a \leq M \frac{(1+\epsilon)}{(1-\epsilon)} \leq M(1 + C\epsilon)$ for $\epsilon \leq c$.

Similar lower bound
$\Rightarrow$ w.p. $\geq 1 - 2\exp(-k)$

$(1 - C\epsilon) M \leq \|z\| \leq M(1 + C\epsilon)$ $\forall$ $z \in E \cap S^{n-1}$

Absorb C into def of k to finish proof
QED

Geometric form of DMT

Recall: One-to-one correspondence between norms and symmetric convex bodies (in $\mathbb{R}^n$).

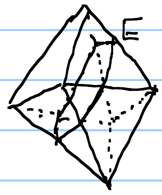
$\|\cdot\|_X \longrightarrow B_X \xleftarrow{\text{unit ball}}$, $K \rightarrow \|\cdot\|_K := \inf\{\lambda > 0 : x \in \lambda K\}$
$\uparrow$ convex body , $\uparrow$ convex body

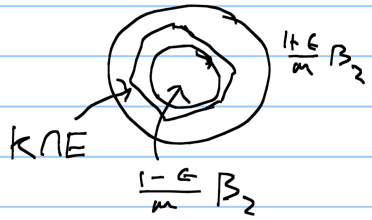
- Given convex body K, apply DMT for $\|\,\|_K$.

Assumption: $\|x\|_K \leq b\|x\|_2 \iff K \supseteq \frac{1}{b} B_2^n$

Conclusion: $(1-\epsilon) M\|X\|_2 \leq \|X\|_K \leq (1+\epsilon)M\|X\|_2$

$\iff \frac{1-\epsilon}{M} B_2^n \cap E \subseteq K \cap E \subseteq \frac{1+\epsilon}{M} B_2^n \cap E$





Thm (DMT, random sections) Let K be a symmetric convex body s.t. $K \supseteq \frac{1}{b} B_2^n$. Let

$$M = \mathbb{E}\|X\|_K \qquad X \sim \text{unif}(S^{n-1}).$$

Then, $\forall \epsilon > 0$, a random subspace $E \subseteq G_{n,k}$ of dimension $k = c_\epsilon \left(\frac{M}{b}\right)^2 n$ satisfies w.p. $\geq 1 - 2e^{-k}$ that

$$\frac{(1-\epsilon)}{M} B_2^n \cap E \subseteq K \subseteq \frac{1+\epsilon}{M} B_2^n \cap E$$

Random projection version of DMT

Theme: projections & intersections onto subspaces are in dual relationship.

Recall: Given a convex body K, the polar body is

$$\boxed{K^\circ := \{ x \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \quad \forall y \in K \}}$$

- $\|z\|_{K^\circ} := \inf\{\lambda > 0 : z \in \lambda K^\circ\} = \sup_{y \in K^\circ} \langle z, y \rangle$
- $\|\ \|_K$ & $\|\cdot\|_{K^\circ}$ are dual norms

Properties:

- $K \subseteq L \Rightarrow K^\circ \supseteq L^\circ$
- $(aK)^\circ = \frac{1}{a} K^\circ$
- K convex, closed $\Rightarrow K^{\circ\circ} = K$ (bipolar thm)

$\bullet \boxed{(K \cap E)^\circ = P_E K^\circ}$
subspace

$\boxed{\text{Proof: Exercise}}$

Note: Polar is taken in E not $\mathbb{R}^n$

Examine DMT for K°:

Assumptions

- $K^\circ \subseteq \frac{1}{b} B_2^n \iff K \subseteq b B_2^n$
- $M := \mathbb{E}\|X\|_{K^\circ} = \mathbb{E} \sup_{y \in K} \langle X, Y \rangle =: W_s(K) \leftarrow$ spherical mean width

Conclusion:

$$\frac{1-\epsilon}{m} B_2^n \cap E \subseteq K^\circ \cap E \subseteq \frac{1+\epsilon}{m} B_2^n \cap E$$

Take polars of all sides (in E!):

$$\Rightarrow \quad \frac{M}{1-\epsilon} \underbrace{P_E B_2^n}_{= E \cap B_2^n} \supseteq P_E K \supseteq \frac{M}{1+\epsilon} P_E B_2^n$$

$\Rightarrow$

Thm (DMT random projections)

Let $K \subseteq \mathbb{R}^n$ be a symmetric convex body s.t. $K \subseteq b B_2^n$. Let

$$M = W_s(K).$$

Then, $\forall \epsilon > 0$, a random projection P_E onto a subspace E uniformly distributed in the Grassmannian satisfies: Suppose $\dim(E) = k = c_\epsilon \left(\frac{m}{b}\right)^2 n$, then w.p. $\geq 1 - 2e^{-k}$,

$$(1-\epsilon) M B_2^n \cap E \subseteq P_E K \subseteq (1+\epsilon) M B_2^n \cap E$$

Literature: [Vershynin, "Lectures in geometric..."]