Chapman-Kolmogorov eqns

Def. Given a Markov chain $(X_n)_{n \geq 0}$, the transition matrix $P$ satisfies

$$p_{ij} = P(X_{n+1} = j \mid X_n = i)$$

Rmk. Properties of $P$

1. $P$ has as many rows/cols as there are states (possibly $\infty$).

2. $0 \leq p_{ij} \leq 1$, $\sum_j p_{ij} = 1 \quad \forall i$ (rows sum to 1)

Matrices that satisfy 2 are called stochastic matrices.

Q: What is $P(X_{k+n} = j \mid X_k = i)$?

Q: $= \sum_j P(X_n = j \mid X_0 = i)$?

Q: $= \sum_j p_{ij} P(X_{k+n} = j \mid X_k = i)$ by stationarity.

Let $p_{ij}^n = P(X_n = j \mid X_0 = i)$.

$n = 0$. $p_{ij}^0 = P(X_0 = j \mid X_0 = i) = \delta_{ij} \Rightarrow P^0 = I$

$n = 1$. $p_{ij}^1 = p_{ij}$ by def.

$$p_{ij}^{n+m} = P(X_{n+m} = j \mid X_0 = i)$$

$$= \sum_k p_{ik} P(X_{n+m} = j \mid X_n = k, X_0 = i)$$

$$= \sum_k p_{ik} P(X_{n+m} = j \mid X_n = k, X_0 = i) \cdot P(X_n = k \mid X_0 = i)$$

Markov prop.
\[ p_{ij}^{m+n} = \frac{\sum_k p_{ik} p_{kj}}{\sum_k p_{ik} p_{kj}} \]

Chapman-Kolmogorov eqn.

Eg., \( p_{ij}^2 = \frac{\sum_k p_{ik} p_{kj}}{\sum_k p_{ik} p_{kj}} = (p \cdot p)_{ij} = (p^2)_{ij} \)

By induction,

\[ p_{ij}^n = (p \cdot p \cdots p)_{ij} = (p^n)_{ij} \]

In other words, the transition probabilities for taking \( n \) steps are just \( p^n \).

What if we have random initial state, i.e.,

\[ p(X_0 = i) = \alpha_i \quad \sum_i \alpha_i = 1 \quad \alpha = (\alpha_1, \alpha_2, \ldots) \]

Then, \( P(X_n = j) = \frac{\sum_i p(X_n = j | X_0 = i) \cdot p(X_0 = i)}{\sum_i p(X_0 = i)} \cdot p(X_0 = i) \)

\[ = \frac{\sum_i \alpha_i \cdot p_{ij}}{\sum_i \alpha_i} = (\alpha \cdot p^n)_j \]