Sub-Gaussian r.v.'s

Note Title 2015-09-13

Remarks on Hoeffding $\neq$

① Replace X_i w/ $-X_i$ to give

$$P\left(-\sum_{i=1}^n a_i X_i > t\right) \le e^{-t^2/2}$$

$$\Rightarrow P\left(\left|\sum_{i=1}^n a_i X_i\right| > t\right) \le 2 e^{-t^2/2}$$

② Assumption $\sum_i a_i^2 = 1$ can be removed by rescaling, i.e., $\forall \{a_i, i=1,\dots,n\}$,

$$P\left(\left|\sum_{i=1}^n a_i X_i\right| > t\right) \le \exp\left(\frac{-t^2}{2\sum_{i=1}^n a_i^2}\right)$$

$$= \exp\left(\frac{-t^2}{2\,\mathrm{Var}\left(\sum_{i=1}^n a_i X_i\right)}\right)$$

③ Proof techique generalizes, e.g.,

Thm General Hoeffding

$$\boxed{\begin{array}{l}\text{Let } X_1, \dots, X_n \text{ be i.i.d r.v.'s satisfying } \mathbb{E}X_i = 0,\quad c_i \le X_i \le d_i \text{ a.s.} \\ \text{Then } P\left(\left|\sum_{i=1}^n X_i\right| > t\right) \le 2\exp\left(\frac{-t^2}{2\sum_{i=1}^n (d_i - c_i)^2}\right)\end{array}}$$

($\exists$ other dev. ineq. in same spirit)

Q: What is the largest class of r.v.'s s.t. Hoeffding-type $\neq$ holds, i.e.,

(*) $P\left(\left|\sum_{i=1}^{n} a_i X_i\right| > t\right) \lesssim e^{-\frac{t^2}{2}}$ for $\sum a_i^2 = 1$?

<u>Necessary</u> condition:

(**) $P(|X_i| > t) \lesssim e^{-t^2/2}$

i.e., X_i has "sub-Gaussian" tails.

It turns out (**) is sufficient!
(Proof?)

This deserves special attention!

<u>1.3 Sub-Gaussian r.v.'s</u>

Ref. for this section: [Vershynin, Introduction to non-asymptotic random matrix theory]

Some basic properties of Gaussian r.v., $g \sim N(0,1)$

1. Tails: $P(|g| > t) \lesssim e^{-t^2/2}$
2. Moments: $(\mathbb{E}|g|^p)^{1/p} \lesssim \sqrt{p}$ $\quad p \geq 1$
3. Tails': $\mathbb{E}\, e^{g^2/c} \leq e$
4. MGF: $M_g(t) := \mathbb{E}\, e^{gt} = e^{t^2/2}$

Some facts about moments:

- $(\mathbb{E}|X|^p)^{1/p} =: \|X\|_p$ is a norm.

  It define $L_p(\Omega, P) = \{ X \text{ r.v. on } \Omega \text{ s.t. } \|X\|_p < \infty \}$

  Ω: sample space, P: prob. meas.

- $\|X\|_q \geq \|X\|_p$ for $q > p$ (by Hölder's $\neq$)

- $\|X\|_\infty = \operatorname{ess\,sup}|X|$ i.e. $M \geq \|X\|_\infty$ iff $|X| \leq M$ a.s.

How to define a sub-Gaussian r.v.? Consider the following properties for a r.v. X. For some $k_1, k_2, k_3, k_4 \geq 0$:

1. Tails: $P(|X| > t) \leq \exp\left(1 - \frac{t^2}{k_1^2}\right)$ $\quad t \geq 0$
2. Moments: $(\mathbb{E}|X|^p)^{1/p} \leq k_2 \sqrt{p}$ $\quad p \geq 1$
3. Tails': $\mathbb{E} \exp\left(\frac{X^2}{k_3^2}\right) \leq e$
4. MGF: $\mathbb{E} \exp(tX) \leq \exp(t^2 k_4^2)$ $\quad t \in \mathbb{R}$

Lemma Properties 1–3 are equivalent. Moreover, if $\mathbb{E} X = 0$, property 4 is equivalent to 1–3.

Precise meaning of equivalence: $\exists$ an absolute const. C such that prop. i implies prop. j with $K_j \le C \cdot K_i$ for any two $i,j \in \{1,2,3,4\}$

If $\mathbb{E}X = 0$.

$\boxed{\text{Def. A r.v. } X \text{ is sub-Gaussian if it satisfies (1)-(3)}}$

$\boxed{\text{HW Show that if } \mathbb{E}X \neq 0, \text{ property (4) may fail even if (1)-(3) hold.}}$

Proof of lemma:

(1) $\Rightarrow$ (2), Rescale so $K_1 = 1$.

Recall: For a non-neg. r.v. Z,

$$\mathbb{E}Z = \int_0^\infty P(Z \geq u)\, du$$

$$\Rightarrow \mathbb{E}|X|^p = \int_0^\infty P(|X|^p > u)\, du$$

$$= \int_0^\infty P(|X| > t) \cdot p t^{p-1}\, dt \qquad \left(\text{change of var: } t = u^{1/p}\right)$$

$$\leq \int_0^\infty \exp(1 - t^2)\, p\, t^{p-1}\, dt$$

$$= \frac{ep}{2} \Gamma\left(\frac{p}{2}\right) \qquad \Gamma: \text{ Gamma function}$$

$$\le \left(\frac{ep}{2}\right)\left(\frac{p}{2}\right)^{p/2} \qquad \text{(Stirling's formula)}$$

Take p^{th} root $\Rightarrow$ prop. (2).

<u>2 $\Rightarrow$ 3</u> Rescale so $k_2 = 1$.

Taylor series:

$$\mathbb{E}\exp(cX^2) = \sum_{p=0}^{\infty} \frac{\mathbb{E}(cX^2)^p}{p!} \le \sum_{p=0}^{\infty} \frac{c^p}{p!} \sqrt{2p}^{\,2p}$$

$$= \sum_{p=0}^{\infty} \frac{(2c)^p p^p}{p!}$$

$$\le \sum_{p=0}^{\infty} \frac{(2c)^p p^p}{\left(\frac{p}{e}\right)^p}$$

Stirling's formula: $p! \approx \sqrt{2\pi}\left(\frac{p}{e}\right)^p$

$\left(\frac{p}{e}\right)^p \le p! \le p^p$

$$= \sum_{p=0}^{\infty} (2ce)^p$$

$$= 2 \quad \text{if } c = \frac{1}{4e} \text{ (e.g.)}$$

i.e., prop. 3 holds.

<u>3 $\Rightarrow$ 1</u> By rescaling $k_3 = 1$.

"Exponential Chebychev":

$$P(|X| > t) = P\left(e^{X^2} > e^{t^2}\right) \le \mathbb{E}\, e^{X^2} \cdot e^{-t^2}$$

$$\le e\, e^{-t^2} = e^{1-t^2}$$

$\Rightarrow$ prop 1 holds QED (for 1-3)

<u>HW</u> Proof 1-3 are equiv. to 4 when $\mathbb{E}X = 0$