Q: Robins & blackbirds arrive at a feeder according to independent Poisson processes \( \{R(t) : t \geq 0\} \) and \( \{B(t) : t \geq 0\} \) with rates \( r \) & \( b \).

Find

1. \( P(1st \ 2 \ birds \ are \ robins) \)
2. Dist. of total # of birds at time \( t \)
3. Dist. of blackbirds at time \( t \) given total # of birds is \( n \).

A: \( P(\text{first bird is robin}) = P(\text{Exp}(r) < \text{Exp}(b)) = \frac{r}{r+b} \)

By memorylessness,

\[
P(\text{first 2 are robins}) = \left( \frac{r}{r+b} \right)^2
\]

Similarly,

\[
P(\text{RBRR}) = \frac{r}{r+b} \cdot \frac{b}{r+b} \cdot \frac{r}{r+b} \cdot \frac{b}{r+b}
\]

2. \( \{R(t)+B(t) : t \geq 0\} \) is a Poisson process w/ rate \( r+b \). Thus,

\( R(t)+B(t) \sim \text{Poisson}(t(r+b)) \)

3. \( P(B(t) = k \mid R(t)+B(t) = n) = \frac{P(B(t) = k, R(t) = n-k)}{P(R(t)+B(t) = n)} \)

\[
= \frac{\frac{n!}{k!(n-k)!} \cdot \frac{b^k}{(r+b)^n} \cdot e^{-b}}{\frac{b^n}{n!} \cdot e^{-b}} = \frac{n^k (b)^k}{(r+b)^n} \cdot \frac{e^{r-b}}{n!}
\]

so \( \text{Bin}(n, \frac{b}{r+b}) \)
Q: Let $X \sim \text{Poisson}(\lambda)$ be the number of games played between Alice and Bob. Suppose Alice wins each game independently w/ prob $p$.

Let $A$ be the number of games Alice wins. Let $B$ be the number of games Bob wins.

1. What is the dist of $A$?
2. Are $A$ & $B$ independent?

$P(A = k) = \sum_{n=k}^{\infty} P(A = k | X = n) \cdot P(n)$

$= \sum_{n=k}^{\infty} \frac{\lambda^n e^{-\lambda}}{n!} \cdot (1-p)^{n-k} \cdot p^k \cdot \frac{\lambda^k e^{-\lambda}}{k!}$

$= \frac{\lambda^k e^{-\lambda}}{k!} \sum_{n=k}^{\infty} \frac{(1-p)^{n-k}}{(n-k)!} \cdot p^k \cdot \lambda^k e^{-\lambda}$

$= \frac{\lambda^k e^{-\lambda}}{k!} \sum_{n=k}^{\infty} \frac{p^k \cdot \lambda^k e^{-\lambda}}{n!}$

$= \frac{\lambda^k e^{-\lambda}}{k!} \cdot p^k \cdot \lambda^k e^{-\lambda}$

so $A \sim \text{Poisson}(p\lambda)$

2. Independent (hint: Exercise for you)

These properties extend to the case of Poisson processes:

**Prop:** Let $\{N(t): t \geq 0\}$ be a rate $\lambda$ Poisson process. Suppose its events are independently classified into:

- Type 1 w/ prob $p$
- Type 2 w/ prob $1-p$

Let $N_1(t) = \#$ Type 1 events by time $t$

$N_2(t) = \#$ Type 2 events by time $t$

Then $N_1$ & $N_2$ are independent Poisson processes w/ rates $p\lambda$ & $(1-p)\lambda$.
Pf See text.

**Conditional dist. of arrival times**

**Q:** Let bus arrival times be a rate \( \lambda \) Poisson process, \( N \). Suppose 1 bus arrived in \([0, t]\).

(I.e. condition on event \( N(t)=1 \).)

What is the dist of the time it arrived?

**A:** Let \( T_1 = \) first arrival time.

\[
P(T_1 < s | N(t) = 1) = \frac{P(T_1 < s, N(t) = 1)}{P(N(t) = 1)} = \frac{P(N(s) = 1, N(t) - N(s) = 0)}{P(N(t) = 1)}
\]

\[
= \frac{\lambda s \cdot e^{-\lambda s} \cdot e^{-\lambda(t-s)}}{\lambda t \cdot e^{-\lambda t}} = \frac{s}{t}
\]

i.e. \( Uniform[0, t] \)