Poisson processes (formal definitions).

**Def 1**
Let \( X_1, X_2, \ldots \sim \text{Exp}(\lambda) \).

\[ X_i = \text{time it takes next event to occur.} \]

Poisson process w/ rate \( \lambda \):
\[ N(t) = \# \text{ of events that have occurred by time } t. \]
i.e. \( N(t) \geq k \iff X_1 + X_2 + \ldots + X_k \leq t \)
\[ \text{kth event occurred by time } t. \]

Second def of Poisson process needs:

**Def (little o notation)** We say a fn \( f \) is \( o(h) \) if \( \lim_{h \to 0} \frac{f(h)}{h} = 0 \).

**Ex)** \( f(h) = h^2 \) so \( \frac{h^2}{h} = h \to 0 \)

**Ex)** \( e^h = 1 + h + o(h) \)

\[ \text{why? Taylor series: } e^h = 1 + h + \frac{h^2}{2} + \frac{h^3}{3!} + \ldots \]

**Ex)** \( e^{h+h} = (1+h+o(h))(1+h) = 1 + 2h + o(h) \)

**Def 2 (Poisson process)** A counting process \( \{N(t); t \geq 0\} \)
\[ \text{is a rate } \lambda \text{ Poisson process if:} \]

(i) \( N(0) = 0 \)

(ii) Independ. increments i.e. \( \frac{N(t)-N(s)}{t-s} \) is indep of
\[ \# \text{ of events in } [s, t] \]
\[
\frac{N(w) - N(u)}{\# \text{ of events in } [u, v]}
\]

For \( t \geq u \geq v \).

(iii) \( P(N(\tau + t) - N(\tau) = 1) = \lambda t + o(t) \)

(iv) \( P(N(\tau + t) - N(\tau) \geq 2) = o(t) \)

Prop: Both definitions are equivalent:

(2) \( N(t) - N(0) \sim \text{Poisson} (\lambda(t-0)) \)

Pf: See text.

Q: Let \( X \sim \text{Poisson}(\lambda) \) and \( Y \sim \text{Poisson}(\beta) \) be independent. What is the distribution of \( X + Y \)?

A: Consider a rate-1 Poisson process \( \{N(t) : t \geq 0\} \).

Let \( X = N(\alpha) - N(\alpha - \lambda) \sim \text{Poisson}(\lambda) \)

\( Y = N(\beta + \alpha) - N(\alpha) \sim \text{Poisson}(\beta + \alpha - \lambda) = \text{Poisson}(\beta) \)

since increments are independent.

\( X + Y = N(\alpha) + N(\beta + \alpha) - N(\alpha) = N(\beta + \alpha) \sim \text{Poisson}(\beta + \alpha) \).

Prop: Let \( \{N_1(t) : t \geq 0\} , \{N_2(t) : t \geq 0\} \) be independent Poisson processes with rates \( \lambda_1 \) and \( \lambda_2 \).

Let \( N(t) = N_1(t) + N_2(t) \). Then \( \{N(t) : t \geq 0\} \) is a rate \( \lambda_1 + \lambda_2 \) Poisson process.

Pf: We need to show that properties (i), (ii), (iii), (iv) of Def 2 hold:

(i): \( N(0) = N_1(0) + N_2(0) = 0 \)
(iii) Write \( \Delta N = N(t + h) - N(t) \)

\[
P(\Delta N = 1) = P(\Delta N_1 + \Delta N_2 = 1) = P(\Delta N_1 = 0) \cdot P(\Delta N_2 = 1) + P(\Delta N_1 = 1) \cdot P(\Delta N_2 = 0)
\]

\[
= \left(1 - P(\Delta N_1 > 0)\right) \cdot P(\Delta N_2 = 1)
+ \left(1 - P(\Delta N_2 > 0)\right) \cdot P(\Delta N_1 = 1)
\]

\[
= \begin{pmatrix}
1 - \lambda_1 \cdot h + o(h) \\
1 - \lambda_2 \cdot h + o(h)
\end{pmatrix}
\begin{pmatrix}
\lambda_2 \cdot h + o(h) \\
\lambda_1 \cdot h + o(h)
\end{pmatrix}
\]

\[
= \lambda_2 h + \lambda_1 h + o(h)
\]

(iii) \( P(\Delta N = 0) = P(\Delta N_1 = 0) \cdot P(\Delta N_2 = 0) = (1 - \lambda_1 \cdot h + o(h))
\]

\[
(1 - \lambda_2 \cdot h + o(h))
\]

\[
= 1 - (\lambda_1 + \lambda_2) h + o(h)
\]

\[\implies P(\Delta N \geq 2) = 1 - P(\Delta N = 1) - P(\Delta N = 0)
\]

\[
= 1 - \left[(\lambda_1 + \lambda_2) h + o(h)\right] - \left[1 - (\lambda_1 + \lambda_2) h + o(h)\right]
\]

\[
= o(h)
\]