I. Problems to be handed in:

1. Let $C$ be a communicating class of a Markov chain. We say that $C$ is closed if $P_{ij} = 0$ for all states $i \in C$ and $j \notin C$. In other words, a communicating class is closed if there is no escape from that class.

   (a) Show that a finite communicating class $C$ is closed if and only if its states are recurrent.
   (b) Find an example of a Markov chain with no closed communicating class.

   **Solution:** (a) Assume first that a finite communicating class $C$ is closed. Then restricting the Markov chain to $C$ is also a Markov chain, which is finite and irreducible. Therefore its states are recurrent. Thus the states of $C$ are recurrent in the original Markov chain, too.

   Now suppose that a finite communicating class $C$ has recurrent states, we prove that $C$ is closed. Assume to the contrary that there are states $i \in C$ and $j \notin C$ with $P_{ij} > 0$. As $i$ is recurrent, there must be a way back from $j$ to $i$, so $i$ and $j$ communicate. But then $i$ and $j$ are in the same communicating class, that is, $j \in C$. This is a contradiction, which proves our claim.

   (b) Consider the Markov chain with state space $\{0, 1, 2, 3, \ldots\}$ and let $P_{i,i+1} = 1$ for all $i \geq 0$. This Markov chain has communicating classes $\{0\}$, $\{1\}$, $\{2\}$, and so on, and none of them is closed.

2. Textbook Chapter 4 Exercise 33.

   **Solution:** Consider the Markov chain such that $X_n$ is the type of the $n$th exam. We can calculate the transition probabilities conditioning on the performance of the class:

   $P_{1,1} = 0.3(1/3) + 0.7(1) = 0.8,$
   $P_{1,2} = P_{1,3} = 0.3(1/3) = 0.1,$
   $P_{2,1} = 0.6(1/3) + 0.4(1) = 0.6,$
   $P_{2,2} = P_{2,3} = 0.6(1/3) = 0.2,$
   $P_{3,1} = 0.9(1/3) + 0.1(1) = 0.4,$
   $P_{3,2} = P_{3,3} = 0.9(1/3) = 0.3.$

   As the Markov chain is irreducible and finite, the long run proportion of state $i$ equals $\pi_i$, the $i$th coordinate of the unique stationary distribution $\pi$ given by $\pi = \pi P$ and $\sum_i \pi_i = 1$. By solving this system we obtain that $\pi_1 = 5/7$ and $\pi_2 = \pi_3 = 1/7$.

3. Textbook Chapter 4 Exercise 36 (a), (b), (c) with $P_{0,0} = 0.3$ and $P_{0,1} = 0.7$.

   **Solution:**
   (a) Conditioning on the state on Tuesday gives
   
   $P(\text{good msg on Tue}) = P(X_1 = 0)P(\text{good msg on Tue} | X_1 = 0) + P(X_1 = 1)P(\text{good msg on Tue} | X_1 = 1) = P_{0,0}p_0 + P_{0,1}p_1 = 0.3p_0 + 0.7p_1.$

   (b) Similarly, conditioning on the state on Friday gives
   
   $P(\text{good msg on Fri}) = P_{0,0}^4p_0 + p_1P_{0,1}^4 = 0.222p_0 + 0.778p_1.$

   (c) The proportion of time being spent in state 0 is $\pi_0$, and fraction $p_0$ of all the messages sent from state 0 are good, so the proportion of time a good message is sent from state 0 is $\pi_0 p_0$. Similarly, the proportion of time a good message is sent from state 1 is $\pi_1 p_1$. Thus the long run proportion of good messages is $\pi_0 p_0 + \pi_1 p_1 = (2/9)p_0 + (7/9)p_1$. 

4. Consider the random walk on the integers \{0, 1, 2, 3\} which takes steps +1 (to the right) with probability \(\frac{1}{3}\) and \(-1\) (to the left) with probability \(\frac{2}{3}\), except at the endpoints where there is reflection; this means that a step from 1 to 0 is always followed by a step from 0 to 1, and a step from 2 to 3 is always followed by a step from 3 to 2.

(a) Determine the transition matrix for this Markov chain.
(b) Argue that the chain is time reversible without considering the detailed balance equations.
(c) What is the stationary distribution?
(d) Suppose the Markov chain has been running for a long time. What fraction of time has it spent in state 0?

Solution:
(a) The transition matrix is
\[
P = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & \frac{2}{3} & 0 & \frac{1}{3} \\
1 & \frac{2}{3} & 0 & \frac{1}{3} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
(b) The number of jumps from \(i\) to \(j\) and the number of jumps from \(j\) to \(i\) can differ by no more than one, since after a jump from \(i\) to \(j\) it is only possible to return to \(i\) through \(j\), so between any two jumps from \(i\) to \(j\) there must be a jump from \(j\) to \(i\) and vice versa. Thus the long run proportion of jumps from \(i\) to \(j\) equals the long run proportion of jumps from \(j\) to \(i\) are equal, giving the detailed balance equations \(\pi_i P_{i,j} = \pi_j P_{j,i}\) for any pairs \(i, j\).
(c) The detailed balance equations give give \(\pi_0 = (2/3)\pi_1\), \((1/3)\pi_1 = (2/3)\pi_2\), \((1/3)\pi_2 = \pi_3\), and we have \(\sum_i \pi_i = 1\). Solving the system we get \(\pi_0 = 2/7\).
(d) As the chain is irreducible and finite, the long run proportion of time spent in state 0 is given by \(\pi_0 = 2/7\).

5. A bishop starts at the bottom left of a chess board and performs random moves. At each stage, she picks one of the available legal moves with equal probability, independently of the earlier moves. Let \(X_n\) be her position after \(n\) moves. What is the mean number of moves before she returns to her starting square?

Hint: Read and apply Example 4.36.

Solution:
The black squares of the chessboard are vertices of a connected graph, where two vertices are connected by an edge if a bishop can move from one corresponding square to another. In each square \(i\) write the degree of the vertex \(i\), that is, the number of edges starting from \(i\). The table below shows the degree of squares the bishop can reach.

\[
\begin{array}{ccccccccc}
7 & 7 & 7 & 7 & 7 & 7 & 9 & 11 & 11 & 9 \\
7 & 9 & 9 & 9 & 7 & 11 & 13 & 9 & 13 & 11 \\
7 & 11 & 11 & 9 & 7 & 13 & 9 & 7 & 7 & 7
\end{array}
\]

By Example 4.36, \(\pi_i\) is proportionate to the degree of \(i\) for any vertex \(i\), that is, \(\pi_i = \frac{\deg(i)}{\sum_j \deg(j)}\). (We apply the Example such that each edge has weight one, which corresponds to the random walk on the graph.) Thus if 0 denotes the left bottom corner, then we have
\[
\pi_0 = \frac{7}{7 \cdot 14 + 9 \cdot 10 + 11 \cdot 6 + 13 \cdot 2} = \frac{1}{40}.
\]
6. Let \( X_n \) be a Markov chain with state space \( \{0, 1, 2, \ldots \} \) with transition probabilities

\[
p_{i,i+1} = a_i \quad \text{and} \quad p_{i,0} = 1 - a_i
\]

for all state \( i \), where \( a_i \) are numbers between 0 and 1. Let \( b_0 = 1 \) and \( b_i = a_0 a_1 \cdots a_i \). Show that the chain is

(a) recurrent if and only if \( \lim_{i \to \infty} b_i = 0 \),
(b) positive recurrent if and only if \( \sum_{i=0}^{\infty} b_i < \infty \).
(c) Find the stationary distribution in the positive recurrent case.

Solution: For the sake of simplicity let us change the condition \( b_0 = 1 \) to \( b_{-1} = 1 \) (so \( b_0 = a_0 \)), this does not change the statements.

(a) We calculate the probability that we never return to 0. Let \( A_n \) the we never returned after \( n \) transitions, that is, none of \( X_1, \ldots, X_n \) is 0. This is only possible if \( X_1, \ldots, X_n = n \), so \( P(A_n) = a_0 \cdots a_{n-1} = b_{n-1} \). We never return to 0 if and only if all events \( A_n \) hold. Since \( A_{n+1} \subseteq A_n \) for all \( n \), we have

\[
P(\text{no return to 0}) = P(\cap_{n=1}^{\infty} A_n) = \lim_{n \to \infty} P(A_n) = \lim_{n \to \infty} b_n.
\]

Thus 0 recurrent if and only if \( \lim_{n \to \infty} b_n = 0 \).

Remark: In the trivial case when one of the \( a_i \)'s is 0, the chain is not irreducible, so recurrence and positive recurrence is not defined. This makes the problem a bit imprecise, but we can fix it by considering the (positive) recurrence of state 0 in this case.

(b) Let \( M_0 \) denote the time of first return to 0. Then for all \( n \geq 1 \) we have

\[
\{M_0 = n\} = \{X_i = i \text{ for all } i \leq n-1, X_n = 0\},
\]

thus

\[
P(M_0 = n) = a_0 \cdots a_{n-2}(1 - a_{n-1}) = b_{n-2}(1 - a_{n-1}) = b_{n-2} - b_{n-1}.
\]

Thus

\[
E(M_0) = \sum_{n=1}^{\infty} nP(M_0 = n) = \sum_{n=1}^{\infty} n(b_{n-2} - b_{n-1}) = \sum_{n=1}^{\infty} b_n(n + 2 - (n + 1)) = \sum_{n=1}^{\infty} b_n.
\]

So 0 is positive recurrent if and only if \( E(M_0) < \infty \), if and only if \( \sum_{n=1}^{\infty} b_n < \infty \).

(c) The equations of stationarity give \( \pi_i = \sum_j \pi_j p_{j,i} \) for all states \( i \), which gives \( \pi_i = a_{i-1} \pi_{i-1} \) for all \( i \geq 1 \) and \( \pi_0 = \sum_{j=1}^{\infty} \pi_j (1 - a_j) \). From the first equations we obtain by induction that \( \pi_i = b_{i-1} \pi_0 \) for all \( i \geq 1 \), and then the equation for \( \pi_0 \) is automatically satisfied. We also have the equation \( \sum_i \pi_i = 1 \), so \( \pi_0(1 + \sum_{n=0}^{\infty} b_n) = \pi_0(\sum_{i=1}^{\infty} b_i) = 1 \). Thus for all \( i \geq 0 \) we have

\[
\pi_i = \frac{b_{i-1}}{\sum_{i=-1}^{\infty} b_i}.
\]

II. Recommended problems: These provide additional practice but are not to be handed in.
Chapter 4: 32, 42, 62, 47, 57, 68b.

7. (a) Prove that periodicity is a class property: If \( i, j \) are communicating states of a Markov chain, then their periods satisfy \( d_i = d_j \). Hint: Show that \( d_i \) divides \( d_j \).
(b) Let \( i \) be a state of a Markov chain. Prove that the following statements are equivalent:

(1) The period of \( i \) is \( d \),
(2) There is a positive integer \( N \) such that for each \( n > N \) we have \( P^n_{i,i} > 0 \) if and only if \( d \) divides \( n \).
Solution:

(a) Assume that \( i \) and \( j \) communicate. Let us denote by \( a \mid b \) that \( a \) divides \( b \). It is enough to prove that \( d_i \mid d_j \), since if \( d_i \) divides all integers in a set, then it divides their greatest common divisor, too. Fix \( k, \ell \geq 1 \) such that \( P_{i,j}^k > 0 \) and \( P_{j,i}^\ell > 0 \) and let \( m = k + \ell \). Then \( P_{i,i}^m = P_{i,i}^{k+\ell} \geq P_{i,j}^k P_{j,i}^\ell > 0 \). Similarly, \( P_{i,i}^{m+n} = P_{i,i}^{k+n+\ell} \geq P_{i,j}^k P_{j,i}^n P_{j,i}^\ell > 0 \). Therefore \( d_i \mid m \) and \( d_i \mid m + n \) by definition. Thus \( d_i \mid (m + \ell) - m = n \), and we are done.

(b) First we prove (1) \( \Rightarrow \) (2). Let \( d \) be the period of \( i \) and let \( I = \{ n \geq 1 : P_{i,i}^n > 0 \} \). As \( I \) is closed under addition (why?), it is enough to prove that there exists a positive integer \( M \) such that for all \( m \geq M \) the number \( dm \) can be written as a linear combination of elements of \( I \) with non-negative integer coefficients. For this statement see Lemma 1.27 in [http://pages.uoregon.edu/dlevin/MARKOV/markovmixing.pdf](http://pages.uoregon.edu/dlevin/MARKOV/markovmixing.pdf).

Now we show (2) \( \Rightarrow \) (1). Let \( p \) be the period of \( i \). As \( p \) divides any number in \( I = \{ n \geq 1 : P_{i,i}^n > 0 \} \), specially we have \( p \mid md \) and \( p \mid (m + 1)d \) for \( m > N \). Thus \( p \mid (m + 1)d - md = d \) as well, so \( p \leq d \).

Assume to the contrary that \( p < d \), then (1) \( \Rightarrow \) (2) implies that \( I \) contains all large enough multiples of \( p \), but a large enough number can be in \( I \) if and only if it is a multiple of \( d \), which is a contradiction.