Math 303 Assignment 2: Due Wednesday, January 24 at start of class

I. Problems to be handed in:

1. Find the Markov chain:

   (a) Find a Markov chain that has an infinite number of states, is irreducible, aperiodic, and null recurrent, i.e., all states are null recurrent. (*Hint: You've seen such a Markov chain in class.*)

   (b) Find a Markov chain that has an infinite number of states, is irreducible, aperiodic, and positive recurrent, i.e., all states are positive recurrent. Show that it has all of these properties.

2. Suppose that a Markov chain is irreducible, and satisfies $P_{i,i} > 0$ for some state i. Show that it must be aperiodic.

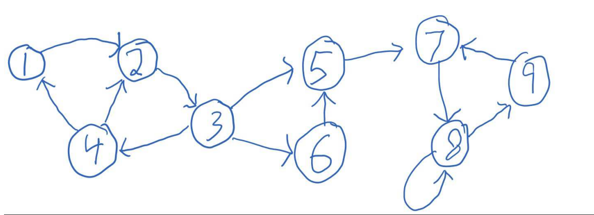
3. Consider a chessboard with standard game play rules.

   (a) Suppose there is a single king making making random moves, i.e., on each move he picks one of the allowable squares to move to uniformly at random. Is the corresponding Markov chain irreducible? Is it aperiodic?

   (b) Same question but with the king replaced by a bishop (recall, bishops move diagonally).

   (c) Same question but with the bishop replaced by a knight (the horsey that goes like an L).

4. Consider a Markov chain with the following transition diagram, where an arrow is included for any nonzero transition probability.



   (a) What are the communicating classes?

   (b) For each communicating class, what is the period? Is the class recurrent?

   (c) How many stationary distributions are there? Why?

5. **Fix that Markov chain:** In applications, one is often interested in the stationary distribution of a Markov chain that may not satisfy all of the requirements needed to converge to stationarity (and for which the dimension is too large to compute eigenvectors). How can this be fixed? To be precise, let P be the transition matrix for an N-state, irreducible, periodic Markov chain. On one hand, the MC is irreducible and has a finite number of states, so it has a unique stationary distribution. On the other hand, it is periodic, so the Markov chain generally does not approach the stationary distribution (or any distribution) in the limit.

   Let us consider instead the matrix $P_\lambda := \lambda I_N + (1-\lambda)P$ for some $\lambda \in (0,1)$, where I_N is the N by N identity matrix. Show that the following hold.

   (a) P_λ is a valid transition matrix, i.e., it is a stochastic matrix.

   (b) It is the transition matrix for an irreducible, aperiodic Markov chain. (Thus, meeting requirements to converge to stationarity.)

   (c) It has the same stationary distribution as P.

6. Textbook Chapter 4 Exercise 38.

II. Recommended problems: These provide additional practice but are not to be handed in. Textbook Chapter 4: Exercises 14, 16, 21.