Math 303 Assignment 1: Due Wednesday, January 17 at start of class

I. Problems to be handed in:

1. A Markov chain X_n with states $0, 1, 2$ has the transition probability matrix

$$\mathbf{P} = \begin{bmatrix} 0.2 & 0.6 & 0.2 \\ 0.3 & 0.3 & 0.4 \\ 0.5 & 0.4 & 0.1 \end{bmatrix}.$$

   If $\mathbb{P}(X_0 = 0) = 0.4$ and $\mathbb{P}(X_0 = 1) = \mathbb{P}(X_0 = 2) = 0.3$, find the distribution of X_3.

2. A total of 5 balls are divided between two urns A and B. A ball is chosen uniformly at random. If it is chosen from urn A then it is placed in urn B with probability $1/2$ and otherwise it is returned to urn A. Similarly, if the ball is chosen from urn B then it is placed in urn A with probability $1/3$ and otherwise it is returned to urn B. Let X_n denote the number of balls in urn A after the nth trial. This defines a Markov chain with state space $\{0, 1, \ldots, 5\}$.

   (a) Determine the transition probabilities and draw the transition diagram.

   (b) Calculate $\mathbb{P}(X_2 = 4 \,|\, X_0 = 5)$.

3. Smith plays 7 chess games one after another, and wins each of them independently with probability 0.6. What is the probability that he wins 3 consecutive games?

   Hint: You can find an appropriate Markov chain with only 4 states.

4. Let X_n be a Markov chain with states $0, 1, \ldots, 9$ and transition probabilities $P_{0,0} = P_{0,1} = P_{9,8} = P_{9,9} = 1/2$ an $P_{i,i} = P_{i,i+1} = P_{i,i-1} = 1/3$ for all $1 \leq i \leq 8$.

   (a) Draw the transition diagram.

   (b) What is the probability that X_1, X_2, X_3, X_4 are all smaller than 3 given that $X_0 = 1$?
   Hint: Create a simpler Markov chain with 4 states.

5. Let us flip a fair coin n times. Calculate the probability p_n such that there are no two consecutive heads.

   Hint: Find a recursion for p_n and solve it.

II. Recommended problems: These provide additional practice but are not to be handed in. Textbook Chapter 4: Exercises 1, 2, 4, 5, 7, 8, 10.