Math 608D, Assignment 2

1. Let $\mathcal{X} \subset \mathbb{R}^n$ be a finite point cloud, i.e., $|\mathcal{X}| = N$. We wish to find a nearly isometric embedding of $\mathcal{X}$ into $\ell^m_1$. To be precise, fix $0 < \epsilon < 1$. We wish to show there is a mapping $A : \mathbb{R}^n \to \mathbb{R}^m$ satisfying

$$(1 - \epsilon) \|x - y\|_2 \leq \|A(x - y)\|_1 \leq (1 + \epsilon) \|x - y\|_2 \quad \text{for all } x, y \in \mathcal{X}.$$ 

What mapping makes this embedding (with high probability)? How small can you choose $m$ as a function of $n, N, \epsilon$?

2. Consider the $p$-norm, $p \geq 1$, $\|x\|_p := \left(\sum_i |x_i|^p\right)^{1/p}$. Let $B^n_p := \{x \in \mathbb{R}^n : \|x\|_p \leq 1\}$ be the unit ball for this norm. Let $N(K, \|\cdot\|_p, \epsilon)$ be the covering number of the set $K \subset \mathbb{R}^n$ using the $p$-norm, that is, the minimal cardinality of a subset $X \subset K$ satisfying: for any $x \in K$ there is $y \in X$ such that $\|x - y\|_p \leq \epsilon$. Bound the following:

(a) $N(B^n_1, \|\cdot\|_1, \epsilon)$.
(b) $N(B^n_p, \|\cdot\|_p, \epsilon)$.
(c) $N(K, \|\cdot\|_\infty, \epsilon)$, where $K \subset B^n_\infty$ is a polytope with $N$ vertices. (Can you beat the bound in the previous problem when $N$ is not too large?)
(d) Tensors. Consider $\mathbb{R}^{n \times n \times n}$, i.e., arrays with 3 modes. (Picture a rubix cube.) We call elements of this set tensors. A rank-1 tensor takes the form $T = u \times v \times w$, where $u, v, w \in \mathbb{R}^n$.

In other words,

$T_{i,j,k} = u_i \cdot v_j \cdot w_k.$

We say that a tensor has rank at most $r$ if it may be written as the sum of $r$ rank-1 tensors.

The Frobenius norm of a tensor is $\|T\|_F = \sqrt{\sum_{i,j,k} T_{i,j,k}^2}$. Let $K$ be the set of rank-$r$ tensors with Frobenius norm at most 1. Bound $N(K, \|\cdot\|_F, \epsilon)$. (I believe such a bound was made in the literature in the past few years, but I haven’t read the details.)

(e) The packing number $P(K, \|\cdot\|_F, \epsilon)$ of a set $K$ is the maximal cardinality of a subset $\mathcal{X} \subset K$ satisfying $\|x - y\|_F > \epsilon$ for all $x, y \in \mathcal{X}$ with $x \neq y$. Lower bound $P(K, \|\cdot\|_F, \epsilon)$ with $K$ as in the last problem. (I’m not sure whether this has been done.)

3. Single index model/1-layer neural net. A 1-layer neural net is a function $f : \mathbb{R}^n \to \mathbb{R}^m$ of the following form. Let $W \in \mathbb{R}^{m \times n}$ be a weight matrix and let $\sigma : \mathbb{R} \to \mathbb{R}$ be some activation function. Then

$$f(x) = \sigma(Wx) \quad \text{where } \sigma \text{ acts elementwise.}$$

In other words, the $i$th entry of $f(x)$ is

$$(f(x))_i = \sigma(W_i, x) \quad \text{where } W_i \text{ is the } i \text{th row of } W.$$ 

Note: The above model is also called the single-index model of statistics.

In this problem we explore the behaviour of $f$ under the assumption that the weights are Gaussian, i.e., assume $W_{i,j} \sim \mathcal{N}(0, 1)$ and all entries are independent. We will check whether $f$ preserves Euclidean norms.
(a) Suppose that $\sigma$ is a rectified linear unit i.e.,

$$\sigma(x) = \max(x, 0).$$

i. Fix $x \in \mathbb{R}^n$. Let $T_x = \|f(x)\|_2$. What is $E T_x^2$?

ii. Bound

$$\|T_x - E T_x\|_2.$$ 

iii. Let $K \subset S^{n-1}$ be a finite set, i.e., $|K| \leq \infty$. Bound

$$E \sup_{x \in K} |T_x - E T_x|.$$ 

iv. We now pass to the infinite case. Bound

$$E \sup_{x \in S^{n-1}} |T_x - E T_x|.$$ 

v. Now the more challenging infinite case. Let $K \subset S^{n-1}$ be an arbitrary infinite set. Bound

$$E \sup_{x \in K} |T_x - E T_x|.$$ 

(Your answer should depend on how “large” $K$ is.)

(b) Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a general function. Complete Items i-v above in this setting. Your answers should depend on properties of $\sigma$. (You may assume what you need.)