Math 302, Assignment 1 Solutions

(1) Let $S = \{1, \{\}, c\}$ be a sample space. List all possible events.

Solution: $\{\}$, $\{1\}$, $\{\{\}\}$, $\{c\}$, $\{1, \{\}\}$, $\{1, c\}$, $\{\{\}, c\}$, $\{1, \{\}, c\}$.

(2) Let Ω be a sample space and $\mathbb{P}$ be a probability. Prove that there can't exist events E, F that satisfy

$$\mathbb{P}(E \setminus F) = \frac{1}{3},\ \mathbb{P}(E \cup F) = \frac{1}{2},\ \text{and } \mathbb{P}\left((E \cap F)^c\right) = \frac{3}{4}.$$

Solution: The formula $\mathbb{P}(A) = 1 - \mathbb{P}(A^c)$ yields that

$$\mathbb{P}(E \cap F) = 1 - \mathbb{P}\left((E \cap F)^c\right) = \frac{1}{4}.$$

The third axiom of probability implies that

$$\frac{1}{2} = \mathbb{P}(E \cup F) = \mathbb{P}(E \setminus F) + \mathbb{P}(E \cap F) + \mathbb{P}(F \setminus E) = \frac{1}{3} + \frac{1}{4} + \mathbb{P}(F \setminus E),$$

but $\mathbb{P}(F \setminus E) \geq 0$, which gives a contradiction.

(3) We roll a fair die until the first 1 comes up. What is the probability that the number of tosses is odd?

Solution: Let $X \sim \text{Geom}(1/6)$.

$$\begin{aligned}\mathbb{P}(X \in \{1, 3, 5, \ldots\}) &= \frac{1}{6} + \left(\frac{5}{6}\right)^2 \cdot \frac{1}{6} + \left(\frac{5}{6}\right)^4 \cdot \frac{1}{6} + \ldots \\ &= \frac{1}{6} \cdot \frac{1}{1 - 25/36} = \frac{6}{11}.\end{aligned}$$

In the second equality, we summed the geometric series.

(4) Assuming a fair poker deal, what is the probability of a
 (a) royal flush
 (b) straight flush
 (c) flush
 (d) straight
 (e) two pair

Solution: The number of all poker hands is $\binom{52}{5} = 2{,}598{,}960$ and they are equally likely.

a) In a given suit there is only one royal flush, there are 4 possible suits, so the probability is

$$\frac{4}{2{,}598{,}960} = \frac{1}{649{,}740}.$$

b) In a given suit there are 10 sequences of five in a row, since the lowest value can be $A, 2, 3, 4, 5, 6, 7, 8, 9, 10$, but the last one is a royal flush. As there are 4 possible suits, the probability is

$$\frac{4 \cdot 9}{2{,}598{,}960} \approx \frac{1}{72{,}193}.$$

c) A given suit contains 13 cards, there are $\binom{13}{5} = 1{,}287$ ways to choose 5. There are 10 which form a sequence, so the probability is

$$\frac{4 \cdot (1287 - 10)}{2{,}598{,}960} \approx \frac{1}{509}.$$

d) There are 10 possibilities to choose the lowest value of the sequence, then for any value 4 different suit can be assigned. That is $10 \cdot 4^5 = 10{,}240$ possibilities, from which there are 40 straight flushes and royal flushes. Therefore the probability is

$$\frac{10{,}240 - 40}{2{,}598{,}960} \approx \frac{1}{255}.$$

e) There are $\binom{13}{2} = 78$ possibilities to choose the values of the two pairs, and there is 11 possibility for the value of the last card. We can choose $\binom{4}{2} = 6$ possible suits for both pairs and 4 possible suits for the single card, which gives the probability

$$\frac{78 \cdot 11 \cdot 6^2 \cdot 4}{2{,}598{,}960} \approx \frac{1}{21}.$$

(5) How many ways are there to deal 52 standard playing cards to four players (so that each player gets 13 cards)? Suppose you are world champion in card dealing, and can deal 52 cards in just one second. Compare the time it would need you to deal all possible combinations with the current age of the universe.

Solution: Mathematically, the question describes the set of all functions f ("ways of dealing") from the set of cards $C = \{1, \ldots, 52\}$ to the set of players $\{1, 2, 3, 4\}$, in such a way that every player gets exactly 13 cards, i.e. $\#\{i \text{ s.t. } f(i) = 1\} = 13 = \#\{i \text{ s.t. } f(i) = 2\} = \ldots$. Note that we labeled the cards $1, \ldots, 52$ and the players $1, 2, 3, 4$ in an arbitrary way. This does not affect the number of functions from $\{1, \ldots, 52\}$ to $\{1, 2, 3, 4\}$, though, since sets are not ordered.
Clearly, any "way of dealing" f is uniquely defined by the four sets $\{i \text{ s.t. } f(i) = 1\}, \ldots, \{i \text{ s.t. } f(i) = 4\}$, and thus the number of such functions is exactly the number of ways of partitioning $C = \{1, \ldots, 52\}$ into 4 sets, each of size 52. This is just the multinomial coefficient

$$\binom{52}{13, 13, 13, 13} = \frac{52!}{13!^4} = 5.4 \cdot 10^{28}.$$

If you can deal one set of cards per second, you'd be able to deal $3.1 \cdot 10^7$ sets per year. So it would take you $1.7 \cdot 10^{21}$ years to deal all combinations. The current age of the universe is $1.3 \cdot 10^{10}$ years. It would take you about 130 billion times longer to complete your dealing.

(6) We toss a fair die four times. What is the probability that all tosses produce different outcomes?

Solution: The natural sample space S consists of the (ordered) sequences of 4 tosses, because then every outcome is equally likely. Then $|S| = 6^4 =$

1296. The outcomes with different values correspond the 4-permutations of the 6-element set $\{1, \ldots, 6\}$, their number is $6 \cdot 5 \cdot 4 \cdot 3 = 360$. (There are 6 options for choosing the first value, 5 for the second and so on.) Thus the probability is $\frac{360}{1296} = \frac{5}{18}$.

(7) Prove that the number of unordered sequences of length k with elements from a set X of size n is $\binom{n+k-1}{k}$.
Hint: For illustration, first consider the example $n = 4, k = 6$. Let the 4 elements of the set X be denoted a, b, c, d. Argue that any unordered sequence of size 6 consisting of elements a, b, c, d can be represented uniquely by a symbol similar to "$\cdot\cdot|\cdot|\cdot\cdot|\cdot$", corresponding to the sequence $aabccd$. Now count the number of choices for the vertical bars.
Solution: Let the elements of X be $\{x_1, \ldots, x_n\}$. We want to count all sequences $(y_1, \ldots, y_k)$ with $y_i \in X$, not counting as different sequences that are obtained from each other via a permutation of the y_i's.
Any sequence $(y_1, \ldots, y_k)$ is therefore equivalent to precisely one where all (if any) x_1's appear first, then all (if any) x_2's and so on. Let's call a sequence ordered in this way a representative sequence. No two different representative sequences are permutations of each other, and so the number of unordered sequences is exactly equal to the number of representative sequences.
But each representative sequence corresponds uniquely to a symbol of the kind $\cdots|\cdots|\cdots|\cdots$, with exactly k dots, and $n-1$ bars. Namely, the number of dots before the first bar corresponds to the number of x_1's in the representative sequence, the number of dots between the first and the second bar to the number of x_2's in the sequence, and so on.
Since the symbol $\cdots|\cdots|\cdots|\cdots$ has length number of dots + number of bars $= k + n - 1$, and we may select any of these positions for the $n-1$ bars. The number of such choices is $\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$, proving the claim.

(8) You own n colors, and want to use them to color 6 objects. For each object, you randomly choose one of the colors. How large does n have to be so that odds are that no two objects will have the same color (i.e., every object is colored in a different color)?

Solution: A coloring is a map that associates to each object its color. If there are 6 objects and n colors, it is thus defined uniquely by a map from $\{1, 2, 3, 4, 5, 6\}$ to $\{1, 2, \ldots, n\}$. There are n^6 such functions, and thus n^6 ways to color 6 objects with n colors randomly.
The event that no two objects receive the same color corresponds to the set of *injective* functions. There are $n(n-1)\cdots(n-5) = \frac{n!}{(n-6)!}$ of them. Thus, the probability of coloring all objects differently is $p_n = \frac{n!}{n^6(n-6)!}$. The numbers p_n become larger than $\frac{1}{2}$ as soon as $n \geq 24$.

(9) Assume that the events E_1, E_2 are independent.
a) Prove that the events E_1^c, E_2^c are also independent.
b) If, in addition, $\mathbb{P}(E_1) = \frac{1}{2}$ and $P(E_2) = \frac{1}{3}$. Prove that

$$\mathbb{P}(E_1 \cup E_2) = \frac{2}{3}$$

c) If, in addition, E_3 is a third event that is independent of E_1 and of E_2, and such that $\mathbb{P}(E_3) = \frac{1}{4}$. Prove that

$$\frac{17}{24} \leq \mathbb{P}(A \cup B \cup C) \leq \frac{19}{24}.$$

Solution:

(a) We have

$$\begin{aligned}
\mathbb{P}(E_1^c \cap E_2^c) &= 1 - \mathbb{P}(E_1 \cup E_2) \\
&= 1 - \Big(\mathbb{P}(E_1) + \mathbb{P}(E_2) - \mathbb{P}(E_1 \cap E_2)\Big) \\
&= 1 - \Big(\mathbb{P}(E_1) + \mathbb{P}(E_2) - \mathbb{P}(E_1)\mathbb{P}(E_2)\Big) \\
&= (1 - \mathbb{P}(E_1))(1 - \mathbb{P}(E_2)) = \mathbb{P}(E_1^c)\mathbb{P}(E_2^c)
\end{aligned}$$

which proves independence.

(b) There are two ways to do the computation. We may either use inclusion-exclusion according to

$$\begin{aligned}
\mathbb{P}(E_1 \cup E_2) &= \mathbb{P}(E_1) + \mathbb{P}(E_2) - \mathbb{P}(E_1 \cap E_2) \\
&= \mathbb{P}(E_1) + \mathbb{P}(E_2) - \mathbb{P}(E_1)\mathbb{P}(E_2) = \frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{2}{3},
\end{aligned}$$

or we may use that, by (a), E_1^c, E_2^c are also independent, which allows us to compute

$$\mathbb{P}(E_1 \cup E_2) = 1 - \mathbb{P}(E_1^c \cap E_2^c) = 1 - \mathbb{P}(E_1^c)\mathbb{P}(E_2^c) = 1 - \frac{1}{2} \cdot \frac{2}{3} = \frac{2}{3}.$$

(c) Using inclusion/exclusion, we compute

$$\begin{aligned}
\mathbb{P}(E_1 \cup E_2 \cup E_3) &= \mathbb{P}(E_1) + \mathbb{P}(E_2) + \mathbb{P}(E_3) - \mathbb{P}(E_1 \cap E_2) - \mathbb{P}(E_2 \cap E_3) \\
&\qquad - \mathbb{P}(E_1 \cap E_3) + \mathbb{P}(E_1 \cap E_2 \cap E_3) \\
&= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{6} - \frac{1}{12} - \frac{1}{8} + \mathbb{P}(E_1 \cap E_2 \cap E_3) \\
&= \frac{17}{24} + \mathbb{P}(E_1 \cap E_2 \cap E_3)
\end{aligned}$$

We used here that the pairs (E_1, E_2), (E_1, E_3) and (E_2, E_3) are independent. Since we do not assume that the triple E_1, E_2, E_3 is independent, we cannot compute the probability of the $E_1 \cap E_2 \cap E_3$ exactly, but since $E_1 \cap E_2 \cap E_3$ is contained in all three of $E_1 \cap E_2$, $E_1 \cap E_3$ and $E_2 \cap E_3$, it must have probability smaller than any of these, i.e. $0 \leq \mathbb{P}(E_1 \cap E_2 \cap E_3) \leq \frac{1}{12}$. This immediately gives the claim.

(10) Eight rooks are placed randomly on a chess board. What is the probability that none of the rooks can capture any of the other rooks? (In non-chess terms: Randomly pick 8 unit squares from an 8×8 square grid. What is the probability that no two squares share a row or a column?)
Hint: How many choices do you have to place rooks in the first row? After you have made your choice, how many choices do you have for the second? Continue this reasoning.

Solution: The total number of choosing 8 positions for the rooks on a board with 64 fields is $\binom{64}{8}$. The number of favorable outcomes is 8!: there are 8 possibilities two choose a square from the first row, 7 ways to choose one from the second row, and so on. Thus our probability in question is

$$P = \frac{8!}{\binom{64}{8}} = \frac{(8!)^2}{64 \cdots 57}.$$

(11) We toss two dice. Consider the events
E: The sum of the outcomes is even.
F: At least one outcome is 5.
Calculate the conditional probabilities $\mathbb{P}(E \mid F)$ and $\mathbb{P}(F \mid E)$.

Answer:$\mathbb{P}(E) = 1/2$, $\mathbb{P}(F) = 11/36$, $\mathbb{P}(E \cap F) = 5/36$, so

$$\mathbb{P}(E \mid F) = \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)} = 5/11 \text{ and } \mathbb{P}(F \mid E) = \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(E)} = 5/18.$$

(12) A fair die is rolled repeatedly.
(a) Give an expression for the probability that the first five rolls give a three at most two times.
(b) Calculate the probability that the first three does not appear before the fifth roll.
(c) Calculate the probability that the first three appears before the twentieth roll, but not before the fifth roll.

Solution:
a) Let X denote the number of threes on the first five rolls, then our probability is

$$\mathbb{P}(X \leq 2) = \mathbb{P}(X = 0) + \mathbb{P}(X = 1) + \mathbb{P}(X = 2) = (\tfrac{5}{6})^5 + 5(\tfrac{5}{6})^4(\tfrac{1}{6}) + 10(\tfrac{5}{6})^3(\tfrac{1}{6})^2.$$

b) The event in question is that none of the first four rolls is a three. On each die the probability of not rolling three is 5/6, so by independence the probability in question is $(5/6)^4$.

c) Let A be the event that none of the first four rolls is a three, and B be the event that some of the rolls from 5–19 is a three, then our event in question is $A \cap B$. By part b) we have $\mathbb{P}(B) = (5/6)^4$ and similarly we obtain $\mathbb{P}(B^c) = (5/6)^{15}$, so $\mathbb{P}(B) = 1 - (5/6)^{15}$. Since A and B are independent, we obtain

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B) = (5/6)^4 \left(1 - (5/6)^{15}\right).$$

(13) * Let the sequence of events $E_1, E_2, \ldots, E_n$ be independent, and assume that $\mathbb{P}(E_i) = \frac{1}{i+1}$. Show that $\mathbb{P}(E_1 \cup \cdots \cup E_n) = \frac{n}{n+1}$

Solution: The sequence $E_1^c, E_2^c, \ldots, E_n^c$ is also independent. This was shown for $n = 2$ in problem 3 above, and is a special case of Fact 2.23 in the textbook. It can be proven by induction, using the inclusion/exclusion formula. From this fact, the claim follows by the calculation

$$\begin{aligned}\mathbb{P}(E_1 \cup \cdots \cup E_n) &= 1 - \mathbb{P}(E_1^c \cap \cdots \cap E_n^c) \\ &= 1 - \mathbb{P}(E_1^c) \cdots \mathbb{P}(E_n^c) = 1 - \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdots \frac{n}{n+1} \\ &= 1 - \frac{1}{n+1} = \frac{n}{n+1}.\end{aligned}$$