1. Suppose that $X$ has moment generating function $M_X(t) = \frac{1}{4} e^{-3t} + \frac{1}{2} + \frac{1}{4} e^t$.

(a) Find the mean and variance of $X$ by differentiating the m.g.f. above.
(b) Find the p.m.f. of $X$. Use your expression for the p.m.f. to check your answers from part (a).

2. You have two dice, one with three sides labeled 0, 1, 2 and one with 4 sides, labeled 0, 1, 2, 3. Let $X_1$ be the outcome of rolling the first die, and $X_2$ the outcome of rolling the second. The rolls are independent.

(a) What is the joint p.m.f. of $(X_1, X_2)$?
(b) Let $Y_1 = X_1 \cdot X_2$ and $Y_2 = \max\{X_1, X_2\}$. Make a table for the joint p.m.f. of $(Y_1, Y_2)$.
(c) Are $Y_1, Y_2$ independent?

3. Let $X \sim \text{Exp}(2), Y \sim \text{Unif}([1, 3])$, and assume that $X$ and $Y$ are independent. Calculate $P(Y - X \geq \frac{1}{2})$.

4. The random variables $X, Y$ have joint probability density function

$$f(x, y) = \begin{cases} C e^{-x} e^{-x - 2y} / e^{x-1} & \text{if } x > 0 \text{ and } y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is the value of $C$?
(b) Are $X$ and $Y$ independent?
(c) Find $P(X < Y)$.

5. Let $X_1$ and $X_2$ be two discrete random variables with joint p.m.f. $P(X_1 = k_1, X_2 = k_2)$. Prove the following claims from the lecture:

(a) If $g : \mathbb{R}^2 \to \mathbb{R}$ is a function, then

$$\mathbb{E} g(X_1, X_2) = \sum_{k_1, k_2} g(k_1, k_2) \cdot P(X_1 = k_1, X_2 = k_2).$$

*Hint:* Remember that the left hand side is by definition $\mathbb{E} g(X_1, X_2) = \sum_l l \cdot P(g(X_1, X_2) = l)$, where the sum is over all values of $g(X_1, X_2)$, i.e. over all $l$ such that $l = g(k_1, k_2)$ for some value $k_1$ of $X_1$ and some value $k_2$ of $X_2$.
(b) $\mathbb{E}[X_1 + X_2] = \mathbb{E} X_1 + \mathbb{E} X_2$. *Hint:* Use part (a).

6. Let $X$ and $Y$ be either two independent Poisson RV’s, or two independent Exponential RV’s, with parameters $\mu, \lambda$. Compute the p.m.f. / p.d.f. of $X + Y$.

7. Compute the moment generating functions of the Geom($p$) and the Exp($\lambda$) random variables.
8. **Challenge, not marked** Let $X$ be a continuous random variable with p.d.f. $f(x)$ and $g : \mathbb{R} \to \mathbb{R}$ be a strictly increasing function. Show that the p.d.f. of $g(X)$ equals

$$f_{g(X)}(y) = \frac{f(g^{-1}(y))}{g'(g^{-1}(y))}$$

9. Textbook exercises 5.6 and 5.7.