Math 302, Assignment 3

(1) An evil mathematician has trapped you in a dungeon behind 5 doors. Every door is locked with a keypad, in which you must enter a number between 1 and 6000. You enter one random number (with replacement) into the keypad every second. The lock will open if you enter one of 10 special numbers the evil mathematician has selected for each lock.
  (a) Use the exponential random variable to approximate the probability that it'll take you longer than 20 minutes to open the first door.
  (b) Use the Poisson random variable to approximate the probability that after one hour, you have escaped the dungeon.

(2) Suppose that the continuous RV X has c.d.f. given by

$$F(x) = \begin{cases} 0 & \text{if } x < \sqrt{2} \\ x^2 - 2 & \text{if } \sqrt{2} \le x < \sqrt{3} \\ 1 & \text{if } \sqrt{3} \le x \end{cases}$$

  (a) Find the smallest interval $[a, b]$ such that $\mathbb{P}(a \le X \le b) = 1$.
  (b) Find $\mathbb{P}(2 < X < 3)$.
  (c) Find $\mathbb{P}(X = \frac{3}{2})$.
  (d) Find $\mathbb{P}(1 \le X \le \frac{3}{2})$.
  (e) Find the p.d.f. of X.

(3) (a) Define the function

$$f(x) = \begin{cases} 3x - b & x \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

Show that there is no value of b for which this is the p.d.f. of some RV X.
  (b) Let

$$f(x) = \begin{cases} \frac{1}{2}\cos x & x \in [-b, b] \\ 0 & \text{otherwise} \end{cases}$$

Show that there is exactly one value of b for which this could be the p.d.f. of some RV X.

(4) Suppose a continuous RV X has the c.d.f.

$$F(x) = \begin{cases} c \cdot \arctan x & x > 0 \\ 0 & x \le 0 \end{cases}$$

  (a) What must be the value of c?
  (b) Find the p.d.f. of X.
  (c) Find $\mathbb{E}\,X$.
  (d) Compute $\mathbb{E}\left(\frac{1}{\sqrt{1+X^2}}\right)$.

(5) Let $c > 0$ and $X \sim \text{Unif}[0, c]$. Show that the RV $Y = c - X$ has the same c.d.f. and therefore also the same p.d.f. as X.

(6)* Compute the nth moment of an $\text{Exp}(\lambda)$ random variable.

(7) Let X be a random variable with p.d.f.

$$f(x) = \begin{cases} 2x^{-2} & x > 2 \\ 0 & \text{otherwise} \end{cases}$$

  (a) Compute the c.d.f. of X.
  (b) Find $\mathbb{P}(X > 3 | X < 5)$.

(c) Find the median of X, i.e. the value m such that $\mathbb{P}(X > m) = \mathbb{P}(X \le m)$.
(d) Calculate $\mathbb{E}\sqrt{X}$.

(8) A stick of length ℓ is broken into two pieces at a position $X \sim \mathrm{Unif}[0, \ell]$. Let Y denote the length of the smaller piece.
(a) Calculate the c.d.f. of Y, that is, calculate $\mathbb{P}(Y \le b)$.
(b) Calculate the p.d.f. of Y. Can you identify what kind of random variable Y is?

(9) Let X be an $\mathrm{Exp}(2)$ random variable. Find a number a such that $\{X \in [0, 1]\}$ is independent of $\{X \in [a, 2]\}$.

(10) Let $X \sim \mathcal{N}(2, 4)$ be a normal random variable. Compute:
(a) $\mathbb{P}(X < 6)$.
(b) $\mathbb{P}(X \le 6)$.
(c) $\mathbb{P}(X < 1 | X > -1)$.
(d) $\mathbb{E}\, X^2$
(e) Determine c so that $\mathbb{P}(X > c) = \frac{1}{3}$.
To compute probabilities, use only the values of the c.d.f. of a standard normal random variable found here: `https://en.wikipedia.org/wiki/Standard_normal_table#Cumulative`.

(11) Let $X \sim \mathcal{N}(\mu, \sigma^2)$, and, for $a, b \in \mathbb{R}$, define the random variable $Y = aX + b$. Show that $Y \sim N(a\mu + b, a^2\sigma^2)$.

(12) You randomly throw darts at a dartboard, one dart every second. Suppose that every dart independently hits the dartboard at distance X from the center, where X is a $\mathrm{Unif}[0, 30]$ random variable. Your target, the bullseye, is located around the center and has radius 2.
(a) Suppose you throw darts for 1 minute. Approximate the probability that you score more than 5 bullseye.
(b) Approximate the probability that you throw your first bullseye within half a minute?
(c) Suppose that, every morning for 100 days, you throw darts for half a minute as above. Approximate the probability that you will throw more than 75 bullseye.

(13)* Let X be a standard normal random variable. Compute $\mathbb{E}\, X^n$ for all $n \in \mathbb{N}$.