Math 302, Assignment 3 Solutions

(1) An evil mathematician has trapped you in a dungeon behind 5 doors. Every door is locked with a keypad, in which you must enter a number between 1 and 6000. You can enter one number into the keypad every second. The lock will open if you enter one of 10 special numbers the evil mathematician has randomly selected for each lock.

(a) Use the exponential random variable to approximate the probability that it'll take you longer than 20 minutes to open the first door.

(b) Use the Poisson random variable to approximate the probability that after one hour, you have escaped the dungeon.

Solution: The exponential and Poisson approximations replace this problem by the following: You flip one coin every second, flipping Head opens the door, and the probability of flipping Head is $\frac{10}{6000}$. This approximation would be exact if your strategy was to just randomly enter numbers into the keypad. [Of course, a better strategy would be to try to remember the numbers you have already tried unsuccessfully; in average you'd have to remember 600 numbers before your success, so good luck with that unless you're using a strategy; you can convince yourself that the approximations won't change much in the present case (drawing without replacement from an urn with a large number of balls is approximately the same as drawing with replacement)].

(a) On average, we will have $\lambda = 60 \cdot \frac{10}{6000} = 0.1$ successes per minute. We therefore model the time until the first success (in minutes) by an $\mathrm{Exp}(0.1)$ random variable X. The probability in question is then

$$\mathbb{P}(X > 20) = \int_{20}^{\infty} 0.1e^{-0.1x}\, dx = e^{-0.1\cdot 20} = e^{-2}.$$

(b) We model number of doors opened after one hour by a $\mathrm{Poisson}(60 \cdot \lambda)$ random variable Y. The probability in question is then

$$\mathbb{P}(Y \geq 5) = 1 - \mathbb{P}(Y < 4) = 1 - \Big(1 + 6 + \tfrac{6^2}{2!} + \tfrac{6^3}{3!} + \tfrac{6^4}{4!}\Big)e^{-6} \approx 71\%$$

(2) Suppose that the continuous RV X has c.d.f. given by

$$F(x) = \begin{cases} 0 & \text{if } x < \sqrt{2} \\ x^2 - 2 & \text{if } \sqrt{2} \leq x < \sqrt{3} \\ 1 & \text{if } \sqrt{3} \leq x \end{cases}$$

(a) Find the smallest interval $[a, b]$ such that $\mathbb{P}(a \leq X \leq b) = 1$.

(b) Find $\mathbb{P}(2 < X < 3)$.

(c) Find $\mathbb{P}(X = \frac{3}{2})$.

(d) Find $\mathbb{P}(1 \leq X \leq \frac{3}{2})$.

(e) Find the p.d.f. of X.

Solution: a) Since $\mathbb{P}([a, b]) = F(b) - F(a)$ the smallest such interval is $[\sqrt{2}, \sqrt{3}]$.

b) Since F is constant on the interval $[2, 3]$, we have $\mathbb{P}(2 < X < 3) = 0$.

c) The probability of taking a fixed value is zero for any value. *Note:* An essential feature here is that F is a continuous function, as it should be, being the anti-derivative of a p.d.f.. There are situation where the c.d.f. is not continuous (and the p.d.f. has very strong singularities), and in these situations, single values can have nonzero probability. Such random variables are not part of the 302 curriculum.

d)

$$\begin{aligned}\mathbb{P}(1 \le X \le 3/2) &= \mathbb{P}(X \le 3/2) - \mathbb{P}(X < 1)\\ &= F(3/2) - F(1)\\ &= \left(\frac{9}{4} - 2\right) - 0 = \frac{1}{4}.\end{aligned}$$

e)

$$f(x) = F'(x) = \begin{cases} 2x & \text{if } \sqrt{2} < x \le \sqrt{3},\\ 0 & \text{otherwise.}\end{cases}$$

Note that actually $F(x)$ is not differentiable at $x = \sqrt{2}$ and $\sqrt{3}$, we can define $f(x)$ arbitrarily there, it won't change the integrals.

(3) (a) Define the function

$$f(x) = \begin{cases} 3x - b & x \in [0,1]\\ 0 & \text{otherwise}\end{cases}$$

Show that there is no value of b for which this is the p.d.f. of some RV X.

(b) Let

$$f(x) = \begin{cases} \frac{1}{2}\cos x & x \in [-b,b]\\ 0 & \text{otherwise}\end{cases}$$

Show that there is exactly one value of b for which this could be the p.d.f. of some RV X.

Solution: a) First $f(x) \ge 0$ for all $x \in \mathbb{R}$, so $b \le 0$. We also need $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1$, so

$$1 = \int_0^1 (3x - b)\,\mathrm{d}x = \frac{3}{2} - b.$$

Thus we have $b = \frac{1}{2}$ which does not satisfy $b \le 0$, so f is not a density function for any b.

b) We have

$$\int_{-b}^{b} \tfrac{1}{2}\cos x\; dx = \frac{1}{2}(\sin b - \sin(-b)) = \sin b,$$

and this equals 1 if $b = \frac{\pi}{2} + 2\pi k$, where k is any integer. If $k \neq 0$, then the interval $[-b, b]$ would contain points at which $\cos x$ is negative, which is impossible for a p.d.f.. Thus, only $k = 0$ is allowed, and indeed, f is a nonnegative function and has integral 1 with this choice of b. It could therefore be the p.d.f. of a random variable.

(4) Suppose a continuous RV X has the c.d.f.

$$F(x) = \begin{cases} c \cdot \arctan x & x > 0\\ 0 & x \le 0\end{cases}$$

(a) What must be the value of c?
(b) Find the p.d.f. of X.
(c) Find $\mathbb{E}\,X$.
(d) Compute $\mathbb{E}\big(\frac{1}{\sqrt{1+X^2}}\big)$.

Solution: a) We must have $c = \frac{2}{\pi}$, because then $F(x) \to 1$ as $x \to \infty$

b) Differentiating, we get

$$f(x) = \begin{cases} \frac{2}{\pi} \cdot \frac{1}{1+x^2} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

c) By definition of the expectation, we have

$$\mathbb{E}\, X = \int_0^\infty x \cdot \frac{2}{\pi} \frac{1}{1+x^2}\, dx$$

This integral is divergent, and therefore X does not have a well defined average (it is too spread out). We will learn later in the course how this is to be interpreted.

d)

$$\mathbb{E} \frac{1}{\sqrt{1+X^2}} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{1+x^2}} f(x)\, \mathrm{d}x = \frac{2}{\pi} \int_0^\infty \frac{1}{(1+x^2)^{\frac{3}{2}}}\, \mathrm{d}x = \frac{2}{\pi} \left[\frac{x}{\sqrt{1+x^2}} \right]_0^\infty = \frac{2}{\pi}.$$

(5) Let $c > 0$ and $X \sim \mathrm{Unif}[0, c]$. Show that the RV $Y = c - X$ has the same c.d.f. and therefore also the same p.d.f. as X.

Solution: We have

$$\mathbb{P}(Y \leq b) = \mathbb{P}(X \geq c - b) = \begin{cases} 0 & b \leq 0 \\ \int_{c-b}^{c} \frac{1}{c} = \frac{b}{c} & 0 \leq b \leq c \\ 1 & c \leq b \end{cases},$$

which is the same as $\mathbb{P}(X \leq b)$. Since the p.d.f. is the derivative of the cdf, also the p.d.f.'s of X and Y coincide.

(6)* Compute the nth moment of an $\mathrm{Exp}(\lambda)$ random variable.

Solution: We have

$$\mathbb{E}\, X^n = \int_0^\infty x^n \cdot \lambda e^{-\lambda x}\, dx = \lambda^{-n} \int_0^\infty x^n e^{-x}\, dx$$

Let us call the latter integral I_n. Using integration by parts, we have

$$I_n = \Big[-x^n e^{-x} \Big]_0^\infty + n I_{n-1} = n I_{n-1}$$

if $n \geq 1$. Since $I_0 = 1$ by the normalization property (or computation), we conclude that $I_n = n!$, and $\mathbb{E}\, X^n = \lambda^{-n} n!$.

(7) Let X be a random variable with p.d.f.

$$f(x) = \begin{cases} 2x^{-2} & x > 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Compute the c.d.f. of X.
(b) Find $\mathbb{P}(X > 3 | X < 5)$.
(c) Find the median of X, i.e. the value m such that $\mathbb{P}(X > m) = \mathbb{P}(X \leq m)$.
(d) Calculate $\mathbb{E}\, \sqrt{X}$.

Solution: (a)

$$F(b) = \int_{-\infty}^{b} f(x)\, dx = \begin{cases} 0 & b < 2 \\ \int_2^b 2x^{-2} & b \geq 2 \end{cases} = \begin{cases} 0 & b < 2 \\ 1 - 2b^{-1} & b \geq 2 \end{cases}$$

(b)

$$\mathbb{P}(X > 3 | X < 5) = \frac{\mathbb{P}(\{X > 3\} \cap \{X < 5\})}{\mathbb{P}(X < 5)} = \frac{\mathbb{P}(X \in (3,5))}{\mathbb{P}(X < 5)} = \frac{F(5) - F(3)}{F(5)} = \frac{4}{9}$$

(c) We need to solve $F(m) = \frac{1}{2}$, which gives $m = 4$.
(d)

$$\mathbb{E}\sqrt{X} = \int_{-\infty}^{\infty} \sqrt{x}\, f(x)\, dx = \int_{2}^{\infty} \sqrt{x} 2x^{-2}\, dx = \frac{4}{\sqrt{2}}$$

(8) A stick of length ℓ is broken into two pieces at a position $X \sim \text{Unif}[0, \ell]$. Let Y denote the length of the smaller piece.
(a) Calculate the c.d.f. of Y, that is, calculate $\mathbb{P}(Y \leq b)$.
(b) Calculate the p.d.f. of Y. Can you identify what kind of random variable Y is?

Solution: a) The length of the smaller piece takes values $Y \in [0, \ell/2]$. By geometric considerations, the c.d.f. is

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \leq y) = \mathbb{P}(X \leq y \text{ or } X \geq \ell - y) \\ &= \mathbb{P}(X \leq y) + \mathbb{P}(X \geq \ell - y) \\ &= \int_0^y \frac{1}{\ell}\, \mathrm{d}t + \int_{\ell - y}^{\ell} \frac{1}{\ell}\, \mathrm{d}t = \frac{2y}{\ell}. \end{aligned}$$

Thus

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 0, \\ \frac{2y}{\ell} & \text{if } 0 \leq y \leq \ell/2, \\ 1 & \text{if } y > \ell/2. \end{cases}$$

b) We have

$$f_Y(y) = F_Y'(y) = \begin{cases} 2/\ell & \text{if } 0 \leq y \leq \ell/2, \\ 0 & \text{otherwise .} \end{cases}$$

We recognize that $Y \sim \text{Unif}[0, \ell/2]$

(9) Let X be an Exp(2) random variable. Find a number a such that $\{X \in [0,1]\}$ is independent of $\{X \in [a, 2]\}$.

Solution: If $a < 0$ then all probabilities are the same as in the case $a = 0$, so we may assume that $0 \leq a \leq 1$. We have

$$\mathbb{P}(X \in [0,1]) = F_X(1) - F_X(0) = 1 - e^{-2},$$

and

$$\mathbb{P}(X \in [a, 2]) = F_X(2) - F_X(a) = e^{-2a} - e^{-4}.$$

The probability of intersection is

$$\mathbb{P}(X \in [0,1],\, X \in [a,2]) = \mathbb{P}(X \in [a,1])] = F_X(1) - F_X(a) = e^{-2a} - e^{-2}.$$

The definition of independence gives the equation

$$e^{-2a} - e^{-2} = (1 - e^{-2})(e^{-2a} - e^{-4}),$$

so

$$e^{-2a} = 1 - e^{-2}(1 - e^{-2}),$$

that is,

$$a = -\frac{1}{2}\ln(1 - e^{-2}(1 - e^{-2})) \approx 0.062.$$

(10) Let $X \sim \mathcal{N}(2,4)$ be a normal random variable. Compute:
 (a) $\mathbb{P}(X < 6)$.
 (b) $\mathbb{P}(X \leq 6)$.
 (c) $\mathbb{P}(X < 1 | X > -1)$.
 (d) $\mathbb{E}\, X^2$
 (e) Determine c so that $\mathbb{P}(X > c) = \frac{1}{3}$.

To compute probabilities, use only the values of the c.d.f. of a standard normal random variable found here: `https://en.wikipedia.org/wiki/Standard_normal_table#Cumulative`.

Solution: Let $\Phi(z)$ be the c.d.f. of the standard normal RV. Then the c.d.f. of X is $F_X(x) = \Phi(\frac{x-\mu}{\sigma}) = \Phi(\frac{1}{2}x - 1)$. Note that the table on wikipedia only shows the values of $\Phi(z)$ for $z \geq 0$. The other values have to be obtained from symmetry!
(a)

$$\mathbb{P}(X < 6) = \Phi(\tfrac{6-2}{2}) = 0.97725$$

(b) $\mathbb{P}(X \leq 6) = \mathbb{P}(X < 6) = 0.97725$, since X is a continuous RV.
(c) We have $\mathbb{P}(X \in (-1,1)) = \Phi(-\frac{1}{2}) - \Phi(-\frac{3}{2}) = \Phi(\frac{3}{2}) - \Phi(\frac{1}{2}) = 0.93319 - 0.69146$ (by symmetry) and $\mathbb{P}(X > -1) = 1 - \Phi(-\frac{3}{2}) = \Phi(\frac{3}{2}) = 0.93319$ (again by symmetry), and so $\mathbb{P}(X < 1 | X > -1) = 1 - \frac{0.69146}{0.93319} = 0.25903$.
(d) $\mathbb{E}\, X^2 = \sigma^2 + \mu^2 = 8$.
(e) We have $\Phi(0.43) = \frac{2}{3}$ according to the table, and so $\mathbb{P}(X > 2(0.43+1)) = \frac{1}{3}$, that is, $c = 2.86$.

(11) Let $X \sim \mathcal{N}(\mu, \sigma^2)$, and, for $a, b \in \mathbb{R}$, define the random variable $Y = aX + b$. Show that $Y \sim N(a\mu + b, a^2\sigma^2)$.

Solution: Let $X \sim N(\mu, \sigma^2)$ and let $Y = aX + b$. Then $Z = \frac{X-\mu}{\sigma} \sim N(0, 1)$. The cumulative distribution function of Y is

$$\begin{aligned} F_Y(x) &= \mathbb{P}(Y \le x) = \mathbb{P}(aX + b \le x) \\ &= \mathbb{P}(a(\sigma Z + \mu) + b \le x) \\ &= \mathbb{P}\left(Z \le \frac{x - (a\mu + b)}{a\sigma}\right) \\ &= \Phi\left(\frac{x - (a\mu + b)}{a\sigma}\right). \end{aligned}$$

Thus

$$f_Y(x) = F_Y'(x) = \frac{1}{a\sigma}\varphi\left(\frac{x - (a\mu + b)}{a\sigma}\right),$$

where

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

is the p.d.f. of a standard normal random variable. We obtained that $f_Y(x)$ is the density function of a normal random variable with mean $a\mu + b$ and variance $(a\sigma)^2$, so $Y \sim N(a\mu + b, (a\sigma)^2)$.

(12) You randomly throw darts at a dartboard, one dart every second. Suppose that every dart independently hits the dartboard at distance X from the center, where X is a Unif$[0, 30]$ random variable. Your target, the bullseye, is located around the center and has radius 2.

(a) Suppose you throw darts for 1 minute. Approximate the probability that you score more than 5 bullseye.

(b) Approximate the probability that you throw your first bullseye within half a minute?

(c) Suppose that, every morning for 100 days, you throw darts for half a minute as above. Approximate the probability that you will throw more than 75 bullseye.

Solution: (a) There are 60 attempts, each with probability $\mathbb{P}(\text{Unif}[0, 30] \in [0, 2]) = \frac{1}{15}$. The exact probability is therefore $\mathbb{P}(X > 5)$, where $X \sim \text{Bin}(60, \frac{1}{15})$, i.e.

$$\sum_{k=6} \binom{60}{k} (1/15)^k (14/15)^{60-k} = \frac{\text{some integer with 69 digits}}{\text{some integer with 70 digits}}.$$

We approximate the number of successes by a Poisson(4) - RV Y, and get $\mathbb{P}(Y > 5) = 1 - \frac{643}{15}e^{-4} = 21\%$.

(b) The time X in seconds of the first success is a Geom($\frac{1}{15}$) RV. Thus,

$$\mathbb{P}(X \le 30) = 1 - \left(1 - \tfrac{1}{15}\right)^{30} = \frac{\text{some integer with 35 digits}}{\text{some other integer with 35 digits}} \approx 1 - e^{-\frac{1}{15}\cdot 30} = 1 - e^{-2} \approx 86\%.$$

We could have approximated this time by an Exp($\frac{1}{15}$) RV if we were measuring time in seconds, or by an Exp(4) RV if we were measuring time in minutes. In both cases, we get the probability $1 - e^{-2}$.

(c) According to (b), every day, you'll throw a bullseye with probability $p \approx 86\%$. Thus, the number of bullseye in 100 days is a Bin$(100, p)$ random variable X. The exact probability $\mathbb{P}(X > 75)$ is a ratio of two numbers that don't even fit on a page! By the normal approximation, $\frac{X-86}{\sqrt{100\cdot 0.86\cdot 0.14}}$ is approximately a standard normal RV. Therefore, using the customary continuity correction (optional!)

$$\mathbb{P}(X > 75) = \mathbb{P}\left(\tfrac{X-86}{\sqrt{100\cdot 0.86\cdot 0.14}} > \tfrac{75.5-86}{\sqrt{100\cdot 0.86\cdot 0.14}}\right) \approx \mathbb{P}(Z > -3.03) = \mathbb{P}(Z < 3.03) = 0.99878$$

(13)* Let X be a standard normal random variable. Compute $\mathbb{E}\, X^n$ for all $n \in \mathbb{N}$.

Solution: Let $I_n = \mathbb{E}(Z^n)$, we know that $I_0 = 1$ and $I_1 = 0$. Now let $n \geq 2$, we prove a recursion for I_n. As $\varphi'(x) = -x\varphi(x)$, we can use integration by parts with $f(x) = x^{n-1}$ and $g'(x) = x\varphi(x)$:

$$\begin{aligned}
I_n &= \int_{-\infty}^{\infty} x^n \varphi(x)\, \mathrm{d}x \\
&= \int_{-\infty}^{\infty} x^{n-1}(x\varphi(x))\, \mathrm{d}x \\
&= x^{n-1}(-\varphi(x))|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (n-1)x^{n-2}(-\varphi(x))\, \mathrm{d}x \\
&= 0 + (n-1)\int_{-\infty}^{\infty} x^{n-2}\varphi(x)\, \mathrm{d}x \\
&= (n-1)I_{n-2}.
\end{aligned}$$

Let $k!!$ (k semifactorial) denote the product of positive integers from 1 to k which has the same parity as k, so $k!! = k(k-2)(k-4)\dots$ The above recursion implies that $I_n = (n-1)!!I_1 = 0$ if n is odd, and $I_n = (n-1)!!I_0 = (n-1)!!$ if n is even.
If n is odd, then $x^n\varphi(x)$ is odd and integrable on $(-\infty, \infty)$, which proves directly that $\mathbb{E}(Z^n) = \int_{-\infty}^{\infty} x^n \varphi(x)\, \mathrm{d}x = 0$.