Singular Value Decomposition

Fri, Nov. 22nd/19

Recall: spectral thm, $A \in \mathbb{R}^{n \times n}$, $A = A^T \Rightarrow A$ may be decomposed as ~~$A = VDV^T$~~

$V = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cancel{v} \dots & v_n \\ | & | & & | \end{bmatrix}$

$v_1, \dots, v_n$ are orthobasis for $\mathbb{R}$

*; $A = VDV^T$

$= \sum_{i=1}^{n} \lambda_i v_i v_i^T$

$= \sum_{i=1} \lambda_i \, proj_{v_i}$

Q: In terms of $\lambda_1, \dots, \lambda_n, v_1, \dots, v_n$, what is rank(A), R(A), N(A)?

Recall; $proj_v(x) = \langle x, v \rangle \cdot v$

$= vv^T x$

$= \underbrace{proj_v}_{\text{matrix}} \cdot x$ → ties back to $\sum_{i=1} \lambda_i \, proj_{v_i}$

Answer: Let $\lambda_1, \dots, \lambda_r \neq 0$.

$\lambda_{r+1} = \dots = \lambda_n = 0$ (without loss of generality: WLOG)

Then, we may write:

$A = \sum_{i=1}^{r} \lambda_i v_i v_i^T$

$Av_1 = \sum_{i=1}^{r} \lambda_i v_i v_i^T v_1 = \lambda_1 v_1 + 0 + \dots + 0$

so $v_1 \in R(A)$.

Similarly, $v_2, \dots, v_r \in R(A)$

$\Rightarrow span(v_1, \dots, v_r) \subseteq R(A)$

<u>BUT</u> nothing else since $Ax = \sum_{i=1}^{r} \lambda_i v_i \langle v_i, x \rangle$ = lin. combo of $\{v_i\}$ $\in span(v_1, \dots, v_r)$

i.e. $R(A) \subseteq span(v_1, \dots, v_r)$

$\Rightarrow R(A) = span(v_1, \dots, v_r)$

$rank(A) = r$ = # of non-zero eigvals (λ)

$N(A) = R(A^T)^\perp = R(A)^\perp = span(v_1, \dots, v_r)^\perp$

$= span(v_{r+1}, \dots, v_n)$.

$\because R(A^T)^\perp = R(A)^\perp$ from $A = A^T$

∴ note: whole answer relies on fact that A is <u>symmetric</u> → simple.

proof: rank(A) = rank(D)

↳ ties back to
rank-nullity thm
- rank
- orthonormal basis

Q: How does symmetric matrix A act as a linear transformation?

$$Ax = \sum_{i=1}^{n} \lambda_i v_i v_i^T x = \sum_{i=1}^{n} \lambda_i \, proj_{v_i}(x).$$

write $x = \underline{\alpha_1} v_1 + \underline{\alpha_2} v_2 + \dots + \underline{\alpha_n} v_n$

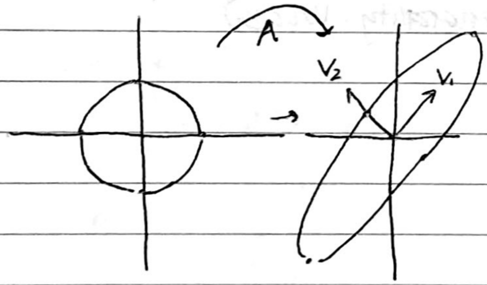
coord w/ respect to $v_1, \dots, v_n$. what does A do?

$Ax = (\lambda_1 \alpha_1) v_1 + (\lambda_2 \alpha_2) v_2 + \dots + (\lambda_n \alpha_n) v_n$ → dilate coordinates.

↳ dilates/rescales coordinates w/ respect to orthobasis $v_1, \dots, v_n$.

A acting on the ball:

$\{ \underbrace{Ax : \|x\|_2 \le 1}_{\text{Euclidian ball}} \}$



$\lambda_1 = 2$, $v_1 \to 2v_1$

$\lambda_2 = \frac{1}{2}$, $v_2 \to \frac{v_2}{2}$

What is badly behaved on a computer? → Wed. lecture.

→ when A is barely invertible; ellipsoid is squished; ex. $\lambda_2 \sim 0$. ex. $\frac{1}{1\,000\,000}$

Introduction to singular value theorem: (SVD)

It turns out any matrix can be decomposed in a similar way.

3 versions: 1) Rank-1 summands version

Let $A \in \mathbb{R}^{m \times n}$. then, A can be decomposed as:

$A = \sum_{i=1}^{r} \sigma_i u_i v_i^T$. | note: σ_i, not λ_i

where $\sigma_1 \ge \sigma_2 \ge \dots \ge \sigma_r > 0$ are called singular values.

$u_1, \dots, u_r \in \mathbb{R}^m$ are orthonormal , | note: $u_i v_i^T$ = rank 1 matrix

$v_1, \dots, v_r \in \mathbb{R}^n$ are othonormal.

and $r = rank(A)$

$v_{r+1}, \dots v_n$ are orthobasis for $N(A)$

$u_{r+1}, \dots u_r$ are orthobasis for $R(A)$.

Sophie's insight:

A acting on a ball.

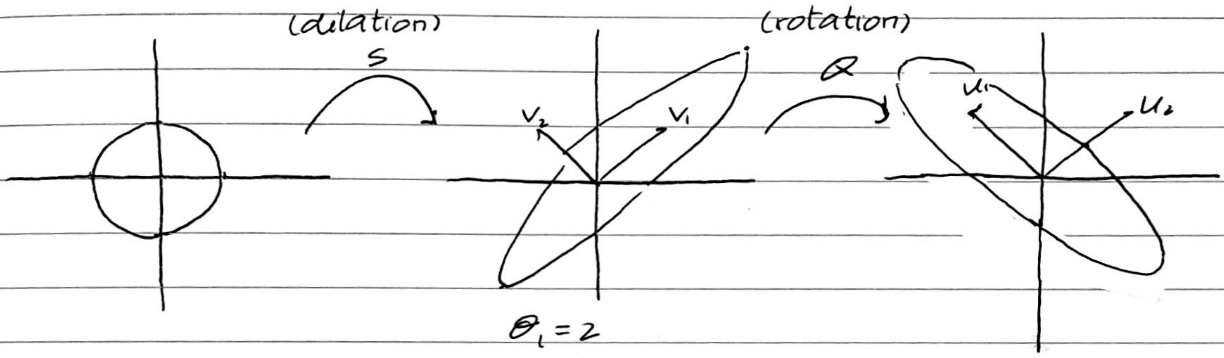
$\therefore$ rank-1 multiplication

$$A = \overset{(*)}{\left(\sum_{i=1}^{r} u_i v_i^T\right)}\left(\sum_{j=1}^{r} \theta_j v_j v_j^T\right)$$

(*) isomorphic matrix from span ($u_i \dots v_i$).

$$= \underbrace{\sum_{i \neq j} \theta_j u_i \underbrace{v_i^T v_j}_{0} v_j^T}_{0} + \sum_{i=j} \theta_i u_i \underbrace{v_i^T v_i}_{1} v_i^T$$

$$A = \sum_i \theta_i u_i v_i^T$$



can be replaced by 1T: $v_1 \to u_1$

$\theta_1 = 2$

$\theta_2 = \frac{1}{2}$

Q $\to$ What do you mean by isomorphism? (Q)

Lemma: Q is an isomorphism from inner product space span ($v_1, \dots v_r$) to inner product space span($u_1, \dots u_r$). I.e. it is bijective, preserves linearity & preserves inner products (thus norms)

$\downarrow$ from span($v_1, \dots, v_r$) to span($u_1, \dots, u_r$)

NOT ambient space $\to$ amb. space

Intuition:

Reminder: $x = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_r$

$Qx = \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_r$ $\to$ same coordinates w/ different orthonormal basis

* Isomorphism:

preserves structure, i.e. inner products (in this case)

application: principal component analysis, by changing rank(A), r based on noise

q's: Is the decomposition unique?

How would you prove that a matrix has a SVD (full rank).