Math 223 Assignment 8

1. Sove the differential equation given the initial conditions $x_1(0) = -3$, $x_2(0) = 4$.

$$\begin{array}{rcrcr} \frac{d}{dt}x_1(t) & = & & + & x_2(t) \\ \frac{d}{dt}x_2(t) & = & -2x_1(t) & - & 2x_2(t) \end{array}$$

2. You are given a 3 dimensional vector space $V \subseteq \mathbf{R}^5$. Could there be a 3×6 (not a typo) matrix A with nullspace of A being V? Explain. Could there be 6×5 matrix B with nullspace of B being V? Explain. In either case, if you were given a basis for the three dimensional space V, how would you find the desired matrix assuming it exists.

3. 3. Consider the two planes π_1: $x - y + 2z = 3$ and π_2: $x + 2y + 3z = 6$.

   a) Find the intersection of π_1 and π_2 in vector parametric form.

   b) What is the angle (or just the cosine of the angle) formed by the two planes? (This is defined as the angle between their normal vectors. A normal vector is a vector orthogonal $u - v$ for every u, v in the plane.)

   c) Find the distance of the point $(-1, 2, 2)$ to the plane π_1. (That is, find the distance between $(-1, 2, 2)$ and the closest point in π_1.)

   d) Find the equation of the plane parallel to π_1 through the point $(3, 2, 0)$.

   e) Imagine the direction $(0, 0, 1)^T$ as pointing straight up from your current position $(0, 0, 0)^T$ in 3-space and the plane π_2 as a physical plane. If a marble is placed on π_2 at the point $(6, 0, 0)^T$, what direction will the marble roll under the influence of gravity?

4. Given a matrix $A \in \mathbf{R}^{n\times n}$, we define the trace

$$\mathrm{tr}(A) = \sum_{i=1}^{n} A_{i,i},$$

   i.e., the sum of the diagonal. This is an important quantity.

   a) Let $A, B \in \mathbf{R}^{n\times n}$. Show that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. *Hint: You may wish to express AB using the dot products between rows of A and columns of B. To be precise, let $u_1, u_2, ..., u_n$ be the columns of A^T (rows of A) and $v_1, v_2, ..., v_n$ be the columns of B. Then $(AB)_{i,j} = u_i \cdot v_j$. You can then show that tr(AB) is the <u>dot product</u> between A^T and B (it's up to you to define this dot product between matrices).*

   b) Suppose that A can be diagonalized as $A = MDM^{-1}$ where D is a diagonal matrix of eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$. Show that

$$\mathrm{tr}(A) = \sum_{i=1}^{n} \lambda_i.$$

   Important note: *The above equality is true even if A cannot be diagonalized. In other words, let $\lambda_1, \lambda_2, ..., \lambda_n$ be the n solutions to the characteristic equation* $\det(A - \lambda I) = 0$. *By the Fundamental Theorem of Algebra, there are always n solutions when counted with multiplicity. These are the eigenvalues of A. Then*

$$\mathrm{tr}(A) = \sum_{i=1}^{n} \lambda_i.$$

   You may use this fact without proof.

5. Let A be a $n \times n$ matrix of real entries satisfying $A^2 = -I$. Show that
   a) A is invertible (or *nonsingular*)
   b) A has no real eigenvalues
   c) n is even
   d) (harder question) $\det(A) = 1$. (Hint: Try using the previous question.)
6. Consider two vectors spaces U, V, subspaces of R^m. Define $U + V = \{\mathbf{u} + \mathbf{v} : \mathbf{u} \in U, \mathbf{v} \in V\}$. (This is called the Minkowski sum.) Show that $U + V$ is a vector space. Now show that

   $\dim(U) + \dim(V) = \dim(U \cap V) + \dim(U + V)$.

   (Hint: if we have an $m \times n$ matrix A then $n = \dim(\text{nullspace}(A)) + \text{rank}(A)$. How should we form A?)