MATH 223 Assignment #6 due Friday March 10.

1. Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be a basis for a vector space V. Then if we define $\mathbf{v}_1 = \mathbf{u}_1 + 2\mathbf{u}_3$, $\mathbf{v}_2 = \mathbf{u}_1 + 2\mathbf{u}_2 + 3\mathbf{u}_3$, $\mathbf{v}_3 = \mathbf{u}_2 - \mathbf{u}_3$, show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ forms a basis for V.

2. (from a test)

$$\text{Let} \quad \mathbf{w}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$$

$$\text{NOTE:} \quad \begin{bmatrix} 1 & 2 & 2 \\ 2 & 5 & 4 \\ 0 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$$

Let $f : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ be the linear transformation satisfying

$$f(\mathbf{w}_1) = \mathbf{w}_2 - \mathbf{w}_3, \quad f(\mathbf{w}_2) = -\mathbf{w}_2 + \mathbf{w}_3, \quad f(\mathbf{w}_3) = \mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3.$$

a) Give the matrix representation of f with respect to the basis $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$.

b) Give the matrix representation of f where the input $\mathbf{x}$, is written with respect to the basis $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ and the output $f(\mathbf{x})$ is written with respect to the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ (the standard basis).

c) Is $\mathbf{w}_1$ in the range of f?

3. (from a test)

$$\text{Let} \quad \mathbf{z}_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{z}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{z}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\text{NOTE:} \quad \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix}$$

Let $T : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ be the linear transformation satisfying

$$T(\mathbf{z}_1) = 2\mathbf{z}_2, \quad T(\mathbf{z}_2) = 2\mathbf{z}_2, \quad T(\mathbf{z}_3) = \mathbf{z}_1 + \mathbf{z}_2.$$

a) Give the matrix representation of T with respect to the basis $\{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3\}$.

b) Give the matrix representation of T with respect to the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ (the standard basis). Give the explicit matrix with integer entries.

c) Give the matrix representing T^2 with respect to the basis $\{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3\}$. What is the rank of the matrix representing T^2 with respect to the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$?

4. Solve the system of differential equations

$$\begin{aligned} \tfrac{d}{dt}x_1(t) &= -5x_1(t) + 6x_2(t) \\ \tfrac{d}{dt}x_2(t) &= -1x_1(t) \end{aligned}$$

First find the general solution for $x_1(t), x_2(t)$ as a function of $x_1(0), x_2(0)$. Given $x_1(0) = 6$ and $x_2(0) = 2$, find the solutions explicitly and compute

$$\lim_{t\to\infty} \frac{x_1(t)}{x_2(t)}$$

.

5. Let U and V be two 5-dimensional subspaces of $\mathbf{R}^9$. Show that there is a nonzero vector in $U \cap V$, the intersection of U and V. You may find it helpful to consider the case of two 2-dimensional subspaces of $\mathbf{R}^3$ first but you won't be able to use ideas of lines and planes in R^9.

6. The following question is from a Putnam exam. This version has some added hints to perhaps make it doable.

   a) Show that it is impossible to have vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{t}$ with

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{u}\mathbf{v}^T + \mathbf{w}\mathbf{t}^T$$

   b) We can encode polynomials in two variables x, y as a matrix so that the polynomial $3x^2y - xy^2 + 5x^3y^3$ is encoded by

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

   Thus the i, j entry corresponds to the coefficient of $x^{i-1}y^{j-1}$. Say we have a polynomial $p(x) = x + 2x^2$ and polynomial $q(y) = 5 - y^2$ then we have $p(x)q(y)$ encoded as

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 5 & 0 & -1 & 0 \\ 10 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} [5 \ 0 \ -1 \ 0\,].$$

   Show that in general that a product of two polynomials $p(x), q(y)$ can be encoded as $\mathbf{u}\mathbf{v}^T$ where $\mathbf{u}$ encodes $p(x)$ and $\mathbf{v}$ encodes $q(y)$. Can there exist polynomials $p(x), q(y), r(x), s(y)$ (each of maximum degree 3) such that $p(x)q(y) + r(x)s(y) = 1 + xy + x^2y^2$?