1. Let $A \in \mathbb{R}^{m \times n}$ have rank 1. Show that there exist non-zero vectors $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ so that $A = xy^T$. (Hint: Try a simple case and also compute $xy^T$ for some simple choices for $x$ and $y$.) (Comment: You could explore how to generalize such a result to higher rank.)

2. Determine bases for the following subspaces of $\mathbb{R}^3$.
   a) the line $x = 5t, y = -2t, z = t$.
   b) all vectors of the form $(a, b, c)^T$ such that $a - 3b = 2c$.

3. Let
   
   $$A = \begin{bmatrix}
   0 & 1 & 1 & 2 & -3 & 1 \\
   0 & 2 & 0 & 6 & -6 & 0 \\
   0 & 3 & 7 & 2 & -9 & 7 \\
   0 & 2 & 2 & 4 & -4 & 3
   \end{bmatrix}$$

   Determine a basis for the column space of $A$ (chosen from columns of $A$) and determine a basis for the row space of $A$. Also give a basis for the nullspace of $A$, namely $\{x \in \mathbb{R}^6 : Ax = 0\}$.

4. Show that the set of all vectors $(b_1, b_2, b_3, b_4)^T$ such that the system below is consistent (i.e. can be solved)

   $$\begin{bmatrix}
   2 & 3 & 1 \\
   4 & 3 & 3 \\
   1 & 3 & 0 \\
   2 & 0 & 2
   \end{bmatrix} \begin{bmatrix}
   b_1 \\
   b_2 \\
   b_3 \\
   b_4
   \end{bmatrix}$$

   is a subspace of $\mathbb{R}^4$. Then find a basis of the subspace.

5. Let $A$ be an $n \times n$ matrix with various eigenvalues including $\lambda$ and $\mu$ with $\lambda \neq \mu$. Let $L, M$ be the eigenspaces associated with eigenvalues $\lambda, \mu$ respectively. (That is, $L$ is the set of all eigenvectors with eigenvalue $\lambda$; $M$ is the set of all eigenvectors with eigenvalue $\mu$.) Let $\{u_1, u_2, \ldots, u_p\}$ be a basis for $L$ and let $\{v_1, v_2, \ldots, v_q\}$ be a basis for $M$. Show that $\{u_1, u_2, \ldots, u_p, v_1, v_2, \ldots, v_q\}$ is a linearly independent set of $p + q$ vectors. (Hint: try $p = 1$ and $q = 1$ to start). (Comment: You could explore the case if there were three different eigenvalues and three bases for the eigenspaces).

6. Let $\mathbb{R}^{n \times n}$ denote the vector space of all $n \times n$ matrices (over $\mathbb{R}$). Consider following transformation $f : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}$

   $$f(A) = A^T.$$ 

   Show that this is a linear transformation.

   We say that a matrix $A$ is symmetric if $A^T = A$ and we say that a matrix $A$ is skew-symmetric if $A^T = -A$.

   a) Warmup question: Give a basis for $\mathbb{R}^{n \times n}$. How many elements are in your basis?

   b) What is the dimension of the eigenspace of eigenvalue 1 for $f$? Explain.

   c) What is the dimension of the eigenspace of eigenvalue -1 for $f$? Explain.

   d) Now use the previous question (and other facts) to show that any $A \in \mathbb{R}^{n \times n}$ is a linear combination of a symmetric matrix and a skew-symmetric matrix (you could show this directly of course but I’m asking you to use linear independence/dimension arguments).