Note: For problems 1,2,3, let \( \mathbb{R} \) be the field associated with each vector space.

1. For each of the following sets, circle \( T \) if it is a vector space (including the case when it is a subspace), and \( F \) if it is not. You do not need to show work for this problem. (The definition of addition and scalar multiplication for these sets follow the standard choices.)

(a) \( \{(b_1, b_2, b_3) \text{ such that } b_1 = 1, b_2, b_3 \in \mathbb{R}\} \) \( T \) \( F \)

(b) \( \{(b_1, b_2, b_3) \text{ such that } 2b_1 - 5b_2 + b_3 = 0, b_1, b_2, b_3 \in \mathbb{R}\} \) \( T \) \( F \)

(c) \( \{(b_1, b_2, b_3) \text{ such that } b_2b_3 = 0, b_1 \in \mathbb{R}\} \) \( T \) \( F \)

(d) \( \{(0,0,0)\} \) \( T \) \( F \)

(e) Infinite sequences \( \{x_i, \ i \geq 1, \text{such that } x_{i+1} \geq x_i\} \). \( T \) \( F \)

(f) The set of matrices \( A \) which satisfy \( A^T = A \). \( T \) \( F \)

(g) The set of invertible matrices. \( T \) \( F \)

(h) The set of 4 by 4 matrices with all eigenvalues greater than or equal to 0. \( T \) \( F \)

(i) The set of polynomials with degree at least 3. \( T \) \( F \)

2. Which of the following are subspaces of the vector space of all functions \( f \) with domain \( \mathbb{R} \) and range contained in \( \mathbb{R} \)?

a) all \( f \) such that \( f(-1) = 0 \).

b) all \( f \) such that \( f(x) \leq 0 \) for all \( x \in \mathbb{R} \).

c) all \( f \) of the form \( f(x) = k_1 + k_2 \sin(x) \) where \( k_1, k_2 \in \mathbb{R} \).

3. Consider the two dimensional vector space \( V = \text{span}(\cos^2(x), \sin^2(x)) \), a subspace of all functions from \( \mathbb{R} \to \mathbb{R} \). Which of the following belong to \( V \) (the argument to show \( f \not\in V \) will be more difficult).

(a) 0  (b) 2  (c) \( 3 + x^2 \)  (d) \( \cos(2x) \)

4. Show that \( 1 \) and \( \sqrt{2} \) are linearly independent when we restrict ourselves to the scalar field \( \mathbb{Q} \), the rational numbers. In other words show that there do not exist 4 integers \( a, b, c, d \) with \( b \neq 0, d \neq 0 \) and not both \( a = 0 \) and \( c = 0 \), which satisfy

\[
\frac{a}{b} \times 1 + \frac{c}{d} \times \sqrt{2} = 0.
\]

5. This is a putnam problem. Let \( A \) be a \( 2013 \times 2014 \) matrix of integer entries such that each row sum is 0 (i.e. \( A\mathbf{1} = \mathbf{0} \) where \( \mathbf{1} \) is the \( 2014 \times 1 \) vector of 1’s and \( \mathbf{0} \) is the \( 2013 \times 1 \) vector of 0’s). Show that \( \det(AA^T) = 2014k^2 \) for some integer \( k \).

Hint: You might find it helpful to form a new square matrix \( B \) from \( A \) by adding a row of 1’s. What is \( \det(BB^T) \)?

6. Let \( V \) be a vector space over a field \( F \). Then, given \( \alpha \in F \) and \( v \in V \) such that \( \alpha v = 0 \), prove that either \( \alpha = 0 \) or \( v = 0 \). (Hint: Check those axioms!)