**MATH 223** Assignment #4 Due Friday February 10 at start of class

Note: For problems 1,2,3, let $\mathbb{R}$ be the field associated with each vector space.

1. For each of the following sets, circle T if it is a vector space (including the case when it is a subspace), and F if it is not. You do not need to show work for this problem. (The definition of addition and scalar multiplication for these sets follow the standard choices.)

   (a) $\{(b_1, b_2, b_3)$ such that $b_1 = 1, b_2, b_3 \in \mathbb{R}\}$ T F

   (b) $\{(b_1, b_2, b_3)$ such that $2b_1 - 5b_2 + b_3 = 0, \quad b_1, b_2, b_3 \in \mathbb{R}\}$ T F

   (c) $\{(b_1, b_2, b_3)$ such that $b_2 b_3 = 0, b_1 \in \mathbb{R}\}$ T F

   (d) $\{(0, 0, 0)\}$ T F

   (e) Infinite sequences $\{x_i, \quad i \geq 1,$ such that $x_{i+1} \geq x_i\}$. T F

   (f) The set of matrices $A$ which satisfy $A^T = A$. T F

   (g) The set of invertible matrices. T F

   (h) The set of 4 by 4 matrices with all eigenvalues greater than or equal to 0. T F

   (i) The set of polynomials with degree at least 3. T F

2. Which of the following are subspaces of the vector space of all functions $f$ with domain $\mathbf{R}$ and range contained in $\mathbf{R}$

   a) all $f$ such that $f(-1) = 0$.

   b) all $f$ such that $f(x) \leq 0$ for all $x \in \mathbf{R}$.

   c) all $f$ of the form $f(x) = k_1 + k_2 \sin(x)$ where $k_1, k_2 \in \mathbf{R}$.

3. Consider the two dimensional vector space $V = \text{span}(\cos^2(x), \sin^2(x))$, a subspace of all functions from $\mathbf{R} \to \mathbf{R}$. Which of the following belong to $V$ (the argument to show $f \notin V$ will be more difficult).

   (a) 0 (b) 2 (c) $3 + x^2$ (d) $\cos(2x)$

4. Show that 1 and $\sqrt{2}$ are linearly independent when we restrict ourselves to the scalar field $\mathbf{Q}$, the rational numbers. In other words show that there do not exist 4 integers $a, b, c, d$ with $b \neq 0$, $d \neq 0$ and not both $a = 0$ and $c = 0$, which satisfy
$$\frac{a}{b} \times 1 + \frac{c}{d} \times \sqrt{2} = 0.$$

5. This is a putnam problem. Let $A$ be a $2013 \times 2014$ matrix of integer entries such that each row sum is 0 (i.e. $A\mathbf{1} = \mathbf{0}$ where $\mathbf{1}$ is the $2014 \times 1$ vector of 1's and $\mathbf{0}$ is the $2013 \times 1$ vector of 0's. Show that $\det(AA^T) = 2014k^2$ for some integer $k$.

   Hint: You might find it helpful to form a new square matrix $B$ from $A$ by adding a row of 1's. What is $\det(BB^T)$?

6. Let $V$ be a vector space over a field $F$. Then, given $\alpha \in F$ and $v \in V$ such that $\alpha v = 0$, prove that either $\alpha = 0$ or $v = 0$. (Hint: Check those axioms!)