Note: For problems 1,2,3, let $\mathbb{R}$ be the field associated with each vector space.

1. For each of the following sets, circle T if it is a vector space (including the case when it is a subspace), and F if it is not. You do not need to show work for this problem. (The definition of addition and scalar multiplication for these sets follow the standard choices.)

(a) $\{(b_1, b_2, b_3) \text{ such that } b_1 = 1, b_2, b_3 \in \mathbb{R}\}$

(b) $\{(b_1, b_2, b_3) \text{ such that } 2b_1 - 5b_2 + b_3 = 0, \ b_1, b_2, b_3 \in \mathbb{R}\}$

(c) $\{(b_1, b_2, b_3) \text{ such that } b_2b_3 = 0, b_1 \in \mathbb{R}\}$

(d) $\{(0, 0, 0)\}$

(e) Infinite sequences $\{x_i, \ i \geq 1, \text{ such that } x_{i+1} \geq x_i\}$.

(f) The set of matrices $A$ which satisfy $A^T = A$.

(g) The set of invertible matrices.

(h) The set of 4 by 4 matrices with all eigenvalues greater than or equal to 0.

(i) The set of polynomials with degree at least 3.

2. Which of the following are subspaces of the vector space of all functions $f$ with domain $\mathbb{R}$ and range contained in $\mathbb{R}$?

a) all $f$ such that $f(-1) = 0$.

b) all $f$ such that $f(x) \leq 0$ for all $x \in \mathbb{R}$.

c) all $f$ of the form $f(x) = k_1 + k_2 \sin(x)$ where $k_1, k_2 \in \mathbb{R}$.

3. Consider the two dimensional vector space $V = \text{span}(\cos^2(x), \sin^2(x))$, a subspace of all functions from $\mathbb{R} \to \mathbb{R}$. Which of the following belong to $V$ (the argument to show $f \notin V$ will be more difficult).

(a) 0  
(b) 2  
(c) $3 + x^2$  
(d) $\cos(2x)$

4. Show that 1 and $\sqrt{2}$ are linearly independent when we restrict ourselves to the scalar field $\mathbb{Q}$, the rational numbers. In other words show that there do not exist 4 integers $a, b, c, d$ with $b \neq 0, d \neq 0$ and not both $a = 0$ and $c = 0$, which satisfy

$$\frac{a}{b} \times 1 + \frac{c}{d} \times \sqrt{2} = 0.$$ 

5. This is a putnam problem. Let $A$ be a 2013 $\times$ 2014 matrix of integer entries such that each row sum is 0 (i.e. $A\mathbf{1} = \mathbf{0}$ where $\mathbf{1}$ is the 2014 $\times$ 1 vector of 1’s and $\mathbf{0}$ is the 2013 $\times$ 1 vector of 0’s. Show that $\det(AA^T) = 2014k^2$ for some integer $k$.

Hint: You might find it helpful to form a new square matrix $B$ from $A$ by adding a row of 1’s. What is $\det(BB^T)$?

6. Let $V$ be a vector space over a field $F$. Then, given $\alpha \in F$ and $v \in V$ such that $\alpha v = 0$, prove that either $\alpha = 0$ or $v = 0$. (Hint: Check those axioms!)