1. Compute

\[\begin{align*}
&i) \text{ det } \begin{bmatrix} x & 0 & 0 \\ 10 & x & 0 \\ 52 & 233 & 1 \end{bmatrix}, \\
&ii) \text{ det } \begin{bmatrix} 99 & 100 & 101 \\ 0 & 0 & 0 \\ 4 & e & 98 \end{bmatrix}, \\
&iii) \text{ det } \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}
\end{align*}\]

2. On the midterm you may be expected to compute the eigenvalues and associated eigenvectors for a $3 \times 3$ matrix. We did the $2 \times 2$ case in class. You’ll need to figure out how this generalizes.

Let

\[A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -2 & 2 \\ -1 & 2 & 1 \end{bmatrix}\]

a) Find the eigenvectors of eigenvalue 2. (This should be just Gaussian Elimination)

b) Compute $\text{det}(A - \lambda I)$ (a cubic polynomial in $\lambda$) by the expansion method. The leading term in the cubic polynomial is $-\lambda^3$. I’d recommend pulling out a factor of -1. A check on your work is that $(\lambda - 2)$ should be a factor of the polynomial (why? because 2 is a root).

c) Factorize $\text{det}(A - \lambda I)$ and determine all eigenvalues and for each eigenvalue, describe the associated set of eigenvectors.

3. Assume $A$ is a $3 \times 3$ matrix, and $M$ is an invertible matrix with $A = MDM^{-1}$, where $D$ is the diagonal matrix

\[D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}\]

Show that $(A - 2I)(A - 3I)(A - 4I) = 0$ where 0 denotes the $3 \times 3$ matrix of 0’s.

4. Let $A = (a_{ij})$ be an $n \times n$ matrix with integral entries such that the diagonal entries are all not divisible by 3 ($a_{ii}$ is not evenly divisible by 3) and all off diagonal entries are divisible by 3 ($a_{ij}$ is divisible by 3 for $i \neq j$). Show that $\text{A}$ has $\text{det}(A) \neq 0$, i.e. $A$ is invertible.

5. We know that the determinant is a function $f : \mathbb{R}^{n \times n} \to \mathbb{R}$, satisfying:

(a) $f(AB) = f(A) \cdot f(B)$.

(b) $f(A) \neq 0$ if and only if $A$ is invertible.

Can you find other functions satisfying these requirements? If yes, describe these functions. (Note: Yaniv hasn’t necessarily figured out the most general set of functions as of 9/21/2019. Can you beat the prof?)

6. Are there functions, $f : \mathbb{R}^{n \times n} \to \mathbb{R}$, satisfying the following two requirements instead? If so, describe them.

(a) $f(AB) = f(A) + f(B)$.

(b) $f(A) \neq 0$ if and only if $A$ is invertible.
7. Let $A$ be an $n$ by $n$ matrix and suppose that $A$ has $n$ distinct eigenvalues. For simplicity, assume they are ordered and positive real numbers so that $\lambda_1 > \lambda_2 > \lambda_3 > \ldots > \lambda_n \geq 0$. Suppose the eigenvalues are associated to the eigenvectors $v_1, v_2, \ldots, v_n$ (in matching order). Let $M$ be the matrix whose $i$-th column is $v_i$. Note we have $AM = MD$ where $D$ is the diagonal matrix with the eigenvalues on the diagonal.

(a) Must $M$ be invertible? If so, prove it. If not, give an example where it is not. *Hint: It may be helpful to use the fact that*

$$Mx = x_1v_1 + x_2v_2 + \ldots + x_nv_n, \quad \text{where} \quad x = [x_1, x_2, \ldots, x_n]^T.$$  

(b) If $M$ is invertible, note that $A$ can be diagonalized as

$$A = MDM^{-1}.$$  

In that case, what is the relationship between det$(A)$ and the eigenvalues of $A$?