MATH 223 Assignment #3 Due Friday February 3 at start of class (two pages)

1. Compute

$$i)\ \det\begin{bmatrix} x & 0 & 0 \\ 10 & x & 0 \\ 52 & 223 & 1 \end{bmatrix}, \quad ii)\ \det\begin{bmatrix} 99 & 100 & 101 \\ 0 & 0 & 0 \\ 4 & e & 98 \end{bmatrix}, \quad iii)\ \det\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

2. On the midterm you may be expected to compute the eigenvalues and associated eigenvectors for a 3×3 matrix. We did the 2×2 case in class. You'll need to figure out how this generalizes. Let

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -2 & 2 \\ -1 & 2 & 1 \end{bmatrix}$$

a) Find the eigenvectors of eigenvalue 2. (This should be just Gaussian Elimination)

b) Compute $\det(A - \lambda I)$ (a cubic polynomial in λ) by the expansion method. The leading term in the cubic polynomial is $-\lambda^3$. I'd recommend pulling out a factor of -1. A check on your work is that $(\lambda - 2)$ should be a factor of the polynomial (why? because 2 is a root).

c) Factorize $\det(A - \lambda I)$ and determine all eigenvalues and for each eigenvalue, describe the associated set of eigenvectors.

3. Assume A is a 3×3 matrix, and M is an invertible matrix with $A = MDM^{-1}$, where D is the diagonal matrix

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Show that $(A - 2I)(A - 3I)(A - 4I) = 0$ where 0 denotes the 3×3 matrix of 0's.

4. Let $A = (a_{ij})$ be an $n \times n$ matrix with integral entries such that the diagonal entries are all not divisible by 3 (a_{ii} is not evenly divisible by 3) and all off diagonal entries are divisible by 3 (a_{ij} is divisible by 3 for $i \neq j$). Show that A has $\det(A) \neq 0$, i.e. A is invertible.

5. We know that the determinant is a function $f : \mathbb{R}^{n\times n} \to \mathbb{R}$, satisfying:

   (a) $f(AB) = f(A) \cdot f(B)$.

   (b) $f(A) \neq 0$ if and only if A is invertible.

   Can you find other functions satisfying these requirements? If yes, describe these functions. (Note: Yaniv hasn't necessarily figured out the most general set of functions as of 9/21/2019. Can you beat the prof?)

6. Are there functions, $f : \mathbb{R}^{n\times n} \to \mathbb{R}$, satisfying the following two requirements instead? If so, describe them.

   (a) $f(AB) = f(A) + f(B)$.

   (b) $f(A) \neq 0$ if and only if A is invertible.

7. Let A be an n by n matrix and suppose that A has n distinct eigenvalues. For simplicity, assume they are ordered and positive real numbers so that $\lambda_1 > \lambda_2 > \lambda_3 > ... > \lambda_n \geq 0$. Suppose the eigenvalues are associated to the eigenvectors $v_1, v_2, ..., v_n$ (in matching order). Let M be the matrix whose i-th column is v_i. Note we have $AM = MD$ where D is the diagonal matrix with the eigenvalues on the diagonal.

   (a) Must M be invertible? If so, prove it. If not, give an example where it is not. *Hint: It may be helpful to use the fact that*

   $$Mx = x_1 v_1 + x_2 v_2 + ... + x_n v_n, \qquad \text{where} \quad x = [x_1, x_2, ..., x_n]^T.$$

   (b) If M is invertible, note that A can be *diagonalized* as

   $$A = MDM^{-1}.$$

   In that case, what is the relationship between $\det(A)$ and the eigenvalues of A?