1. Let $A_1 = \begin{bmatrix} 5 & -6 \\ 1 & 0 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$. For each of these two matrices, determine the eigenvalues and for each eigenvalue determine an eigenvector. For $A_2$ the eigenvalues are a little more complicated making the computations a little harder. Then give the diagonalization of each matrix; namely an invertible matrix $M$ and a diagonal matrix $D$ with $AM = MD$. (The equation $AM = MD$ is important because it will yield $A = MDM^{-1}$ and $M^{-1}AM = D$).

2. Let $A$ be a $2 \times 2$ matrix with two different eigenvalues $\lambda_1, \lambda_2$ and associated eigenvectors $v_1, v_2$. Let $v = av_1 + bv_2$. Assume that $|\lambda_1| > |\lambda_2|$. Show that

$$\lim_{n \to \infty} \frac{A^n v}{\lambda_1^n} = av_1.$$ 

How do you define the limit? For $a \neq 0$ this means that we see the eigenvector $v_1$ appearing in the limit.

3. Review the notes on Fibonacci numbers. Let $f_1, f_2$ be two arbitrary integers, not both zero. Consider the sequence $f_1, f_2, f_3, f_4, \ldots$ where $f_i = f_{i-1} + f_{i-2}$ for $i = 3, 4, 5, \ldots$. We wish to show that

$$\lim_{n \to \infty} \frac{f_n}{f_{n-1}} = \frac{1 + \sqrt{5}}{2}.$$ 

Firstly, explain why we can solve for $c_1, c_2$ in the vector equation

$$\begin{bmatrix} f_2 \\ f_1 \end{bmatrix} = c_1 \begin{bmatrix} 1+\sqrt{5} \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1-\sqrt{5} \\ 2 \end{bmatrix}.$$ 

Using our hypothesis that $f_1, f_2$ are not both zero, we deduce that $c_1, c_2$ are not both zero. Secondly, use our hypothesis that $f_1, f_2$ are integers, not both zero, to deduce $c_1 \neq 0$. The irrationality of $\sqrt{5}$ (which you need not prove) combined with $c_1, c_2$ being integers is important. Thirdly verify the limit. If you can’t show $c_1 \neq 0$ then you can still proceed assuming $c_1 \neq 0$ to establish this limit.

Hint: use ideas of the previous question.

4. In this question, we explore the behaviour of $A^n$ when $A$ does not have distinct eigenvectors (up to rescaling).

(a) Let $A := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

i. Find all eigenvectors and eigenvalues of $A$.

ii. Give a simple expression for $A^n$.

(b) Consider a set of 2-tuples satisfying

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}, \quad n = 0, 1, 2, \ldots$$

Let $x_0 = y_0 = 1$. Give a relatively simple expression for $x_n + y_n$. 


5. I wish to see the solutions to a system of equations in Parametric Vector Form (or Vector Parametric Form). For example if the set of solutions is:

\[
\begin{align*}
  x_1 &= -3r - 4s - 2t \\
  x_2 &= r \\
  x_3 &= -2s \\
  x_4 &= s \\
  x_5 &= t \\
  x_6 &= 1/3
\end{align*}
\]

for all choices \( r, s, t \in \mathbb{R} \) then we can write the set of solutions in parametric vector form as follows:

\[
\begin{bmatrix}
  x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1/3 \end{bmatrix} + r \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad r, s, t \in \mathbb{R}.
\]

Give the vector parametric form of all solutions to the following system of equations:

\[
\begin{align*}
  2x_1 + 4x_4 + 6x_5 &= 14 \\
  2x_1 + 5x_4 + 7x_5 &= 16 \\
  3x_1 + 2x_2 + 8x_4 + 9x_5 &= 27 \\
  3x_1 + 4x_2 + 13x_4 + 12x_5 &= 39
\end{align*}
\]

6. Give the solutions in vector parametric form for the plane \( \pi = \{(x, y, z) : 2x - 2y + 3z = 5\} \).

7. Express the inverse of the following matrix \( A \) as a product of elementary matrices.

\[
A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}
\]