1. [20 marks] Using Gaussian Elimination, give all the solutions to the following system of equations in vector parametric form

\[
\begin{align*}
2x_1 + 4x_2 + 2x_3 + 6x_4 - 2x_5 &= 10 \\
4x_1 + 8x_2 + 7x_3 + 15x_4 - 10x_5 &= 26 \\
2x_1 + 4x_2 - 4x_3 + 10x_5 &= -2
\end{align*}
\]

2. Given

\[
A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ -2 & 2 & 4 \end{bmatrix}
\]

a) [10 marks] Determine the eigenvalues for \(A\). Hint: 2 is an eigenvalue.

b) [15 marks] Determine an eigenvector for each eigenvalue. If there is a repeated eigenvalue (a repeated root), determine two eigenvectors which are not multiples of one another.

3. [10 marks] Assume \(A\) satisfies \(AM = MD\) where

\[
A = \begin{bmatrix} 24 & 10 \\ -30 & -11 \end{bmatrix}, M = \begin{bmatrix} 1 & -2 \\ -2 & 3 \end{bmatrix}, D = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}.
\]

If we set

\[
C = \begin{bmatrix} 4^{3/2} & 0 \\ 0 & 9^{3/2} \end{bmatrix},
\]

then \(C^2 = D^3\). Use this fact to find an explicit matrix \(B\) so that \(B^2 = A^3\) (I want the entries explicitly). Explain (displaying \(B\) is insufficient since checking this directly seems difficult).

4. [12 marks] For what values of the variable \(x\) does the matrix product \(AB\) have an inverse when \(A\) and \(B\) are given as follows?

\[
A = \begin{bmatrix} x & 17 & 15 \\ 0 & 2 & 20 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & x & 1 \\ x & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}
\]

5. [5 marks] We have a function \(f : \mathbb{R}^2 \rightarrow \mathbb{R}^2\) with

\[
\begin{align*}
f\left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) &= \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \\
f\left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) &= \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \\
f\left( \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) &= \begin{bmatrix} 5 \\ 8 \end{bmatrix}.
\end{align*}
\]

Can \(f\) be a linear transformation? If yes, what is the associated matrix. If no, give a reason.

6. [8 marks] Assume \(A, B\) are two \(3 \times 3\) matrices with the property that there exist an invertible matrix \(M\) and two diagonal matrices \(D_A, D_B\) so that \(AM = MD_A\) and \(BM = MD_B\). Show that \(AB = BA\).

7. [10 marks] Consider a system of 4 equations. Consider the following row operation that someone has proposed: For the system of equations, replace the old equation 1 with the sum of the old
equation 1 and \(-1\) times the old equation 2 and simultaneously replacing the old equation 2 with the sum of the old equation 2 and \(-1\) times the old equation 1. Does this preserve the set of solutions to the system of equations?

8. [10 marks] Let \(A\) be a \(4 \times 4\) matrix with an eigenvalue of 2. Show that \(A^2 - 3A + 2I\) is not invertible.