MATH 223: A Putnam Problem using our Linear Algebra.

(This problem came from a 1994 Putnam Problem.)

Problem: Let A,B, be two integer 2×2 matrices (i.e. 2×2 matrices with integer entries). Assume they have the property that A, $A+B$, $A+2B$, $A+3B$, $A+4B$ which are integer 2×2 matrices all with integer 2×2 inverse matrices. We are asked to show that $A+5B$ has an integer inverse.

Proof: An integer 2×2 matric C has an integer inverse if and only if $\det(C)=\pm 1$. This isn't too hard to prove using our expression for the inverse.

$$C=\begin{bmatrix} a & b \\ c & d \end{bmatrix}; \qquad C^{-1}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}=\begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}.$$

Since C has integer entries, $\det(C)$ is an integer. In order for C to have an integer inverse, we would need $ad-bc$ to divide all of a,b,c,d. Let $ad-bc=e$. Then we may write $a=ep$, $b=eq$, $c=er$, $d=es$, for some integers p,q,r,s. Then $\det(C)=e^2\det(\begin{bmatrix} p & q \\ r & s \end{bmatrix})$ which is a contradiction unless $e^2=\pm e$ and so $\det(C)=\pm 1$.

Our hypothesis that $A+iB$ has an integer inverse for $i=0,1,2,3,4$ means $\det(A+iB)=\pm 1$ for $i=0,1,2,3,4$.

Now $\det(A+\lambda B)$ is a quadratic expression in λ (write it out!). Moreover it takes on the values ± 1 for 5 values of λ and hence for three values of λ it takes on the same value; say 1 without loss of generality.

A quadratic expression in λ that takes on the same value for 3 choices of λ is a constant. You may know this fact from your knowledge of the functions $f(x)=ax^2+bx+c$. It is actually a consequence of the deep theorem, called the Fundamental Theorem of Algebra. In our case we have $f(x_1)=f(x_2)=f(x_3)$ and so the quadratic function $g(x)=ax^2+bx+(c-f(x_1))$ has three roots at x_1,x_2,x_3 and so must be the zero function. Hence $\det(A+\lambda B)=1$ for all λ or $\det(A+\lambda B)=-1$ for all λ. But then $\det(A+5B)=\pm 1$ and so $A+5B$ has an integer inverse.

The hypotheses are not vacuous. An example of a pair A,B would be

$$A=\begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}, \qquad B=\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$