Big new concepts in MATH 223 include a *vector space*, *linear independence* (or *linear dependence*), and *dimension*.

Definition *A set* $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ *of* k *vectors is said to be* linearly dependent *if there are coefficients* $a_1, a_2, \ldots, a_k$ not all zero *such that* $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = \mathbf{0}$.

Definition*A set* $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ *of* k *vectors is said to be* linearly independent *if when there are coefficients* $a_1, a_2, \ldots, a_k$ *such that* $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = \mathbf{0}$ *then* $a_1 = a_2 = \cdots = a_k = 0$.

Note that $\mathbf{0}$ is a linearly dependent set, since $1 \cdot \mathbf{0} = \mathbf{0}$.

These definitions are more symmetric than for example identifying S as linearly dependent of one vector in S is a linear combination of the others. Note however if $\mathbf{v}_i$ is a linear combination of the other vectors in S, then $\text{span}(S \backslash \mathbf{v}_i) = \text{span}(S)$.

Determining Linear Independence for n-tuples is a problem for Gaussian Elimination. Let $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ as follows

$$V = \text{span}\left\{ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} \right\}$$

We might note that $\mathbf{v}_4 = 2\mathbf{v}_1 + \mathbf{v}_3$. Thu s $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. But such clever observations can be discovered by Gaussian Elimination.

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} = \mathbf{0}$$

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 1 & 3 & 5 & 7 \\ 0 & 1 & 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By elementary row operations we obtain:

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus the set of solutions are

$$\left\{ s \begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 5 \\ -4 \\ 0 \\ 1 \end{bmatrix} \;:\; s, t \in \mathbf{R} \right\}$$

We deduce that $7\mathbf{v}_1 - 4\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$ and $5\mathbf{v}_1 - 4\mathbf{v}_2 + \mathbf{v}_4 = \mathbf{0}$ from which we have $\mathbf{v}_3 = -7\mathbf{v}_1 + 4\mathbf{v}_2$ and $\mathbf{v}_4 = -5\mathbf{v}_1 + 4\mathbf{v}_2$ and so

$$V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$$

Noting that $\mathbf{v}_1, \mathbf{v}_2$ are linearly independent, we have that $\mathbf{v}_1, \mathbf{v}_2$ is a minimal spanning set for V.

Determining whether functions on a domain D are linearly independent is a bit complicated. We can think of a function f as a tuple with number of entries equal to $|D|$. The entries $f(a), f(b), f(c), \ldots$ would be hopeless to list since typically D is infinite (and likely uncountable). But it does indicate a way to show that $f_1, f_2, \ldots, f_k$ are linearly independent by choosing (carefully) k elements $a_1, a_2, \ldots, a_k \in D$ and showing that the k vectors

$$\begin{bmatrix} f_1(a_1) \\ f_1(a_2) \\ \vdots \\ f_1(a_k) \end{bmatrix}, \quad \begin{bmatrix} f_2(a_1) \\ f_2(a_2) \\ \vdots \\ f_2(a_k) \end{bmatrix}, \quad \ldots, \begin{bmatrix} f_k(a_1) \\ f_k(a_2) \\ \vdots \\ f_k(a_k) \end{bmatrix}$$

are linearly independent. The reverse of showing the k vectors are linearly dependent does not show that $f_1, f_2, \ldots, f_k$ are linearly dependent but it might suggest a dependency to try out. Verifying a linear dependency for functions involves checking equality for all elements of the Domain and so typically involves using properties of the functions. Note that showing that $\sum_{i=1}^{k} x_i f_i = \mathbf{0}$ for some choice of multipliers $x_1, x_2, \ldots, x_k$, requires showing that $\sum_{i=1}^{k} x_1 f_i(x) = 0$ for *all* $x \in D$. Verifying a linear dependency for functions involves checking equality for all elements of the Domain and so typically involves using properties of the functions.

It makes some sense to choose a minimal subset $S' \subseteq S$ with $\text{span}(S') = \text{span}(S)$. Then S' must be linearly independent. You might note that the span of the empty set is naturally defined to be $\{\mathbf{0}\}$. Such boundary cases can be a bit awkward.

Definition *For a vector space V, a* basis *is a linearly independent set of vectors S so that span$(S) = V$.*

There would be two ways to find a basis. Either begin with a spanning set, and reduce if there are any dependencies. Alternatively build the basis from the ground up as a linearly independent set contained in V

Theorem *Any basis for a vector space V has the same cardinality.*

Proof: We let $B_1 = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ and $B_2 = \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_\ell\}$ be two bases for V. Assume that $\ell > k$. Now because B_1 is a basis, then any vector in V is a linear combination of vectors in B_1 and so we may write, without strange names for the coefficients, that

$$\mathbf{v}_j = \sum_{i=1}^{k} a_{ij} \mathbf{u}_i$$

Thus if we let $A = (a_{ij})$ be the matrix with these entries then the jth column of A corresponds to $\mathbf{v}_j$. now because $k < \ell$, then when we solve $A\mathbf{x} = \mathbf{0}$, we will have at most k pivot variables and hence at least $\ell - k > 0$ free variables and hence an $\mathbf{x} \neq \mathbf{0}$ with $A\mathbf{x} = \mathbf{0}$.

We think of $\mathbf{x}$ as yielding a linear combination of the $\mathbf{v}_j$'s yielding the zero vector, which would be a contradiction. Let $\mathbf{x} = (x_1, x_2, \ldots, x_\ell)^T$. Then

$$\sum_{j=1}^{\ell} x_j \mathbf{v}_j = \sum_{j=1}^{\ell} x_j \left(\sum_{i=1}^{k} a_{ij} \mathbf{u}_i \right)$$

$$= \sum_{i=1}^{k} \left(\sum_{j=1}^{\ell} a_{ij} x_j \right) \mathbf{u}_i = \sum_{i=1}^{k} 0 \cdot \mathbf{u}_i = \mathbf{0}$$

This has verified that B_2 is linearly dependent, a contradiction to B_2 being a basis and hence we conclude that $k = \ell$.

Definition The *dimension* of a vector space V is the cardinality of any basis for V.

The dimension of $\mathbf{R}^t$ is t since we can identify a basis of $\mathbf{R}^t$ as $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_t\}$ where $\mathbf{e}_i$ denote the vector with a 1 in the ith coordinate and 0's elsewhere. Any vector space V contained in $\mathbf{R}^t$ has dimension at most t. (How should you show that the dimension is at most t? Assume you have $t + 1$ linear independent vectors in V and derive a contradiction). Thus *dimension* is being used as a piece of mathematical terminology for vector spaces in the context of bases and does not refer some English meaning of dimension (length, width, height?). Maybe we would have been better to have a separate term but this is not standard.