Fibonacci Numbers and 2×2 matrices R. Anstee

The fibonacci numbers $f_1, f_2, f_3, \ldots$ satify

$$f_1 = 1,\ f_2 = 1\ f_i = f_{i-1} + f_{i-2} \text{ for } i = 3, 4, 5, \ldots$$

yielding the sequence $1, 1, 2, 3, 5, 8, 13, 21, \ldots$.
If we let f_n denote the nth fibonacci number we get a matrix equation:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f_i \\ f_{i-1} \end{bmatrix} = \begin{bmatrix} f_{i+1} \\ f_i \end{bmatrix}.$$

Thus, if we let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, we can compute f_n as the top entry of $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = A^{n-2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. To compute a high power of A, we compute the eigenvalues and eigenvectors. Now $\det(\begin{bmatrix} 1-k & 1 \\ 1 & 0-k \end{bmatrix}) = k^2 - k - 1$, which has roots $k = \frac{1+\sqrt{5}}{2}$ and $k = \frac{1-\sqrt{5}}{2}$.

$$\text{eigenvalue: } \frac{1+\sqrt{5}}{2}, \text{ eigenvector: } \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$$

$$\text{eigenvalue: } \frac{1-\sqrt{5}}{2}, \text{ eigenvector: } \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}.$$

Thus if we let

$$P = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix},$$

we have $A = PDP^{-1}$ and so $A^t = PDP^{-1}PDP^{-1} \cdots PDP^{-1} = PD^tP^{-1}$.

$$A^t = PD^tP^{-1} = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (\frac{1+\sqrt{5}}{2})^t & 0 \\ 0 & (\frac{1-\sqrt{5}}{2})^t \end{bmatrix} \begin{bmatrix} \frac{\sqrt{5}}{5} & \frac{5-\sqrt{5}}{10} \\ \frac{-\sqrt{5}}{5} & \frac{5+\sqrt{5}}{10} \end{bmatrix}$$

$$= \begin{bmatrix} (\frac{\sqrt{5}}{5})(\frac{1+\sqrt{5}}{2})^{t+1} - (\frac{\sqrt{5}}{5})(\frac{1-\sqrt{5}}{2})^{t+1} & (\frac{5-\sqrt{5}}{10})(\frac{1+\sqrt{5}}{2})^{t+1} + (\frac{5+\sqrt{5}}{10})(\frac{1-\sqrt{5}}{2})^{t+1} \\ (\frac{\sqrt{5}}{5})(\frac{1+\sqrt{5}}{2})^{t} - (\frac{\sqrt{5}}{5})(\frac{1-\sqrt{5}}{2})^{t} & (\frac{5-\sqrt{5}}{10})(\frac{1+\sqrt{5}}{2})^{t} + (\frac{5+\sqrt{5}}{10})(\frac{1-\sqrt{5}}{2})^{t} \end{bmatrix}$$

Now using our formula that $\begin{bmatrix} f_n \\ f_{n-1} \end{bmatrix} = A^{n-2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = A^{n-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we obtain:

$$f_n = \left(\frac{\sqrt{5}}{5}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{\sqrt{5}}{5}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Given that $\frac{1-\sqrt{5}}{2} \approx -.6$ and so $\lim_{k\to\infty} \left(\frac{1-\sqrt{5}}{2}\right)^k = 0$. Thus

$$f_n \text{ is the closest integer to } \left(\frac{\sqrt{5}}{5}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n$$