MATH 223: Diagonalization with Eigenvalues and Eigenvectors.
An application to bird populations (Leslie Matrix).

Sample computation

Let

$$A = \begin{bmatrix} .7 & .3 \\ 2 & 0 \end{bmatrix}$$

An application associated with this matrix is a simple model of a growing bird population. Let

$$x_n = \text{ no. of adults in year } n,$$

$$y_n = \text{ no. of juveniles in year } n.$$

We have a matrix equation to represent changes from year to year. We have 30% of the juveniles survive to become adults, 70% of the adults survive a year, and each adult has 2 offspring (juveniles). We have this information summarized in a matrix equation:

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} .7 & .3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}.$$

We deduce, by induction, that

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = A^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

This is a sample of many applications where we wish to know what happens to A^n as $n \longrightarrow \infty$.

Recall our computation of eigenvalues/eigenvectors for this matrix:

First we define an eigenvector $\mathbf{x}$ of eigenvalue λ to be satisfy $A\mathbf{x} = \lambda\mathbf{x}$ and $\mathbf{x} \neq \mathbf{0}$. This is equivalent to $(A - \lambda I)\mathbf{x} = \mathbf{0}$ and $\mathbf{x} \neq \mathbf{0}$. This can only occur by our previous observations when $\det(A - \lambda I) = 0$ and moreover when $\det(A - \lambda I) = 0$ we can find an $\mathbf{x} \neq \mathbf{0}$ with $A\mathbf{x} = \lambda\mathbf{x}$.

$$\begin{aligned} \det(A - \lambda I) &= \det(\begin{bmatrix} .7 - \lambda & .3 \\ 2 & -\lambda \end{bmatrix}) \\ &= (.7 - \lambda)(-\lambda) - .3 \times 2 \\ &= \frac{1}{10}(10\lambda^2 - 7\lambda - 6) \\ &= \frac{1}{10}(5\lambda - 6)(2\lambda + 1) \end{aligned}$$

Thus we have two eigenvalues $\lambda = \frac{6}{5}, \frac{-1}{2}$.

For $\lambda = \frac{6}{5}$, we solve $(A - \frac{6}{5}I)\mathbf{v} = \mathbf{0}$ for $\mathbf{v} \neq \mathbf{0}$:

$$(A - \frac{6}{5}I)\mathbf{v} = \begin{bmatrix} -.5 & .3 \\ 2 & -1.2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The vector $\mathbf{v} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ works as an eigenvalue of A of eigenvalue $\frac{6}{5}$. We check

$$\begin{bmatrix} .7 & .3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 3.6 \\ 6 \end{bmatrix} = \frac{6}{5} \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

For $\lambda = \frac{-1}{2}$, we solve $(A - \frac{-1}{2}I)\mathbf{v} = \mathbf{0}$ for $\mathbf{v} \neq \mathbf{0}$:

$$(A - \frac{-1}{2}I)\mathbf{v} = \begin{bmatrix} 1.2 & .3 \\ 2 & .5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The vector $\mathbf{v} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$ works as an eigenvalue of A of eigenvalue $\frac{-1}{2}$. We check

$$\begin{bmatrix} .7 & .3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -4 \end{bmatrix} = \begin{bmatrix} -.5 \\ 2 \end{bmatrix} = \frac{-1}{2} \begin{bmatrix} 1 \\ -4 \end{bmatrix}.$$

Note that we will always succeed in finding an eigenvector (a non zero vector) assuming our eigenvalue λ has $\det(A - \lambda I) = 0$.

The following idea is important in a variety of contexts in this course. For a matrix A, assume we have two eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ of eigenvalues λ_1, λ_2. Form the matrix

$$M = [\mathbf{v}_1 \, \mathbf{v}_2].$$

We have the matrix equation

$$AM = MD$$

where

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

Now make the assumption that M is invertible. This is a non trivial assumption. For us, it is true as long as $\mathbf{v}_1 \neq k\mathbf{v}_2$ for any k. We can verify this to be true if $\lambda_1 \neq \lambda_2$. Assume $\mathbf{v}_1 = k\mathbf{v}_2$ and get a contradiction:

$$A\mathbf{v}_1 = A(k\mathbf{v}_2) = kA(\mathbf{v}_2) = k\lambda_2\mathbf{v}_2 = \lambda_2\mathbf{v}_1,$$

$$A\mathbf{v}_1 = \lambda_1\mathbf{v}_1.$$

We conclude that $\lambda_2\mathbf{v}_1 = \lambda_1\mathbf{v}_1$, i.e. $(\lambda_1 - \lambda_2)\mathbf{v}_1 = \mathbf{0}$ and so, with $\mathbf{v}_1 \neq \mathbf{0}$, $\lambda_1 - \lambda_2 = 0$ and so $\lambda_1 = \lambda_2$ which is a contradiction. Thus $\mathbf{v}_1 \neq k\mathbf{v}_2$ for any k.

Now

$$AM = MD \text{ means } M^{-1}AM = D \text{ and } A = MDM^{-1}.$$

In our case

$$A = \begin{bmatrix} .7 & .3 \\ 2 & 0 \end{bmatrix}, \qquad M = \begin{bmatrix} 3 & 1 \\ 5 & -4 \end{bmatrix}, \qquad M^{-1} = \begin{bmatrix} \frac{4}{17} & \frac{1}{17} \\ \frac{5}{17} & \frac{-3}{17} \end{bmatrix}, \qquad D = \begin{bmatrix} \frac{6}{5} & 0 \\ 0 & \frac{-1}{2} \end{bmatrix}$$

Now we have $A = MDM^{-1}$ and so

$$A^2 = MDM^{-1}MDM^{-1} = MD(M^{-1}M)DM^{-1} = MD^2M^{-1},$$

$$A^3 = MDM^{-1}MDM^{-1}MDM^{-1} = MD(M^{-1}M)D(M^{-1}M)DM^{-1} = MD^3M^{-1},$$

$$A^n = MD^nM^{-1}.$$

It is straightforward to compute

$$D^n = \begin{bmatrix} (\frac{6}{5})^n & 0 \\ 0 & (\frac{-1}{2})^n \end{bmatrix},$$

hence

$$A^n = \begin{bmatrix} .7 & .3 \\ 2 & 0 \end{bmatrix}^n = \begin{bmatrix} 3 & 1 \\ 5 & -4 \end{bmatrix} \begin{bmatrix} (\frac{6}{5})^n & 0 \\ 0 & (\frac{-1}{2})^n \end{bmatrix} \begin{bmatrix} \frac{4}{17} & \frac{1}{17} \\ \frac{5}{17} & \frac{-3}{17} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{12}{17}(1.2)^n + \frac{5}{17}(-.5)^n & \frac{3}{17}(1.2)^n - \frac{3}{17}(-.5)^n \\ \frac{20}{17}(1.2)^n - \frac{20}{17}(-.5)^n & \frac{5}{17}(1.2)^n + \frac{12}{17}(-.5)^n \end{bmatrix}.$$

Thus

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{12}{17}(1.2)^n + \frac{5}{17}(-.5)^n \\ \frac{20}{17}(1.2)^n - \frac{20}{17}(-.5)^n \end{bmatrix} \approx \begin{bmatrix} \frac{12}{17}(1.2)^n \\ \frac{20}{17}(1.2)^n \end{bmatrix},$$

where we are using the fact that $\lim_{n\to\infty}(-.5)^n = 0$. One aspect of the result is that the population is growing 20% a year and also the ratio of adults to juveniles is approximately $3 : 5$ in a stable population. A ratio sufficiently far from $3 : 5$ would alert the biologist to the likelihood of the population having undergone some environmental disturbance in the recent past.