MATH 223 Systems of Differential Equations including example with Complex Eigenvalues

First consider the system of DE's which we motivated in class using water passing through two tanks while flushing out salt contamination.

$$\begin{array}{rcrcr} y_1'(t) & = & -\frac{1}{10}y_1(t) & + & \frac{1}{40}y_2(t) \\ y_2'(t) & = & \frac{1}{10}y_1(t) & - & \frac{1}{10}y_2(t) \end{array}, \qquad y_1(0)=60, y_2(0)=0$$

$$\frac{d}{dt}\mathbf{y} = A\mathbf{y} \text{ where } A = \begin{bmatrix} -1/10 & 1/40 \\ 1/10 & -1/10 \end{bmatrix}$$

We may compute

$$\begin{bmatrix} -1/10 & 1/40 \\ 1/10 & -1/10 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} -3/20 & 0 \\ 0 & -1/20 \end{bmatrix} \begin{bmatrix} 1/2 & -1/4 \\ 1/2 & 1/4 \end{bmatrix} = MDM^{-1}$$

I offer several solutions, the first using change of variable (change of basis). The second considers the matrix exponential ($\mathbf{y} = e^{At}\mathbf{y}(0)$) and the third solution considers write the final solution as a linear combination of solutions to the DE to satisfy the initial conditions.

Our first idea was to rewrite $\frac{d}{dt}\mathbf{y} = A\mathbf{y}$ as $\frac{d}{dt}\mathbf{y} = MDM^{-1}\mathbf{y}$ and then $M^{-1}\frac{d}{dt}\mathbf{y} = DM^{-1}\mathbf{y}$. Then using the linearity of differentiation, we have $\frac{d}{d}M^{-1}\mathbf{y} = DM^{-1}\mathbf{y}$. We set

$$\mathbf{z} = M^{-1}\mathbf{y} \text{ and so } \mathbf{y} = M\mathbf{z}$$

and obtain the 'easy' system of differential equations

$$\frac{d}{dt}\mathbf{z} = D\mathbf{z}$$

namely

$$\frac{d}{dt}z_1(t) = (-3/20)z_1(t),$$
$$\frac{d}{dt}z_2(t) = (-1/20)z_2(t),$$

which we solve as
$z_1(t) = z_1(0)e^{(-3/20)t}, \quad z_2(t) = z_2(0)e^{(-1/20)t}$.

We may compute $z_1(0), z_2(0)$ using $y_1(0) = 60$ and $y_2(0) = 0$ and $\mathbf{z} = M^{-1}\mathbf{y}$ to obtain $z_1(0) = 30$ and $z_2(0) = 30$. Now we use $\mathbf{y} = M\mathbf{z}$ and obtain

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 30e^{(-3/20)t} \\ 30e^{(-1/20)t} \end{bmatrix} = \begin{bmatrix} 30e^{(-3/20)t} + 30e^{(-1/20)t} \\ -60e^{(-1/20)t} + 60e^{(-1/20)t} \end{bmatrix}$$

This solution technique of changing variables (changing basis) to make the system of differential equations easy to solve (diagonalization) follows our usual pattern.

A second solution involves tackling the problem directly in what first appears a bit unlikely:

$$\frac{d}{dt}\mathbf{y} = A\mathbf{y}, \qquad \mathbf{y} = e^{At}\mathbf{y}(0)$$

Recall that

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$$

$$\frac{d}{dt}e^{At} = 0 + A + A^2t + \frac{1}{2}A^3t^2 + \cdots = Ae^{At}$$

where the derivative has been done entrywise. We note that e^{At} at $t = 0$ is in fact $e^0 = I$ (we are exponentiating the 2×2 zero matrix) and hence $e^{At}\mathbf{y}(0)$ at $t = 0$ is indeed $\mathbf{y}(0)$ as desired.

We have techniques for doing this namely:

$$\mathbf{y} = e^{At}\mathbf{y}(0) = Me^{Dt}M^{-1}\mathbf{y}(0)$$

This technique can be used for non diagonalizable matrices A if we can find a similar matrix B (i.e. $A = MBM^{-1}$) for which e^{Bt} is easy to compute.

A third solution technique is commonly used when solving DE's, we write our solution in vector form in terms of the eigenvectors

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = 30e^{(-3/20)t}\begin{bmatrix} 1 \\ -2 \end{bmatrix} + 30e^{(-1/20)t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

This suggests another solution strategy. We seek solutions of the form

$e^{\lambda t}\mathbf{v}$ with $\frac{d}{dt}e^{\lambda t}\mathbf{v} = \lambda e^{\lambda t}\mathbf{v} = e^{\lambda t}A\mathbf{v}$ by solving $A\mathbf{v} = \lambda\mathbf{v}$ and hence solving for eigenvalues and eigenvectors of A. Then (it needs to be proven!) we write an arbitrary solution to the system of DE's as

$$\mathbf{y} = ae^{(-3/20)t}\begin{bmatrix} 1 \\ -2 \end{bmatrix} + be^{(-1/20)t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and solving for a, b given the values $\mathbf{y}(0)$.

Consider the system

$$\begin{array}{rcrcl} y_1'(t) & = & & + & y_2(t) \\ y_2'(t) & = & -4y_1(t) & + & 4y_2(t) \end{array}, \qquad y_1(0) = 60, y_2(0) = 0$$

We begin by noticing the follwoing (how to obtain this is not part of your course this term)

$$A = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix}, \qquad M = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}, \qquad S = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

We can solve the system of DE's by the matrix exponential idea: $\mathbf{y}(t) = e^{At}\mathbf{y}(0) = Me^{St}M^{-1}\mathbf{y}(0)$

We now consider a system of DE's that has complex eigenvalues. It arises from considering the Differential Equation

$$y'' = -y, \qquad y(0) = 1, y'(0) = 0$$

If we set $y_1(t) = y$ and $y_2(t) = y'$ then we can set

$$\mathbf{y} = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

and then we can write the DE in vector form as

$$\frac{d}{dt}\mathbf{y} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\mathbf{y}$$

We can compute eigenvalues and eigenvectors in the natural way using **C** instead of **R**.

$$\underset{A}{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}} = \underset{M}{\begin{bmatrix} i & -i \\ -1 & -1 \end{bmatrix}} \underset{D}{\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}} \underset{M^{-1}}{\begin{bmatrix} -\frac{1}{2}i & -\frac{1}{2} \\ \frac{1}{2}i & -\frac{1}{2} \end{bmatrix}}$$

We could use either of the three methods from above. We can use our third method above (that follows from our change of basis idea). Let $\mathbf{v}_i$ be an eigenvector of eigenvalue λ_i. Then as solution to the DE, ignoring initial conditions, is

$$\mathbf{y} = e^{\lambda_i}\mathbf{v}_i$$

In order to match the initial conditions, we take the appropriate linear combination of these solutions from eigenvector/eigenvalue pairs. In our case we have

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = ae^{it}\begin{bmatrix} i \\ -1 \end{bmatrix} + be^{-it}\begin{bmatrix} -i \\ -1 \end{bmatrix}$$

We can solve for a, b by setting $t = 0$, noting $e^0 = 1$, to obtain

$$\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = a\begin{bmatrix} i \\ -1 \end{bmatrix} + b\begin{bmatrix} -i \\ -1 \end{bmatrix} = M\begin{bmatrix} a \\ b \end{bmatrix}$$

We then solve for a, b using M^{-1} to obtain

$$\begin{bmatrix} a \\ b \end{bmatrix} = M^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}i & -\frac{1}{2} \\ \frac{1}{2}i & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}i \\ \frac{1}{2}i \end{bmatrix}$$

We then can compute the solution.

Once, in a previous version of 223, I solved this by substituting

$$e^{it} = \cos(t) + i\sin(t), \quad e^{-it} = \cos(-t) + i\sin(-t) = \cos(t) - i\sin(t)$$

Then I proceeded to solve for a, b which made things much more complicated. Setting $t = 0$ first and then solving for a, b makes things easier. This is easier for computations; both methods spit out an answer. The solution becomes

$$\mathbf{y} = -\frac{1}{2}i(\cos(t) + i\sin(t))\begin{bmatrix} i \\ -1 \end{bmatrix} + \frac{1}{2}i(\cos(t) - i\sin(t))\begin{bmatrix} -i \\ -1 \end{bmatrix} = \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix}$$

Thus the solution to our DE as expected is $y = \cos(t)$ which has $y(0) = 1$ and $y'(0) = 0$.

We can make some additional simplifications to save work. Let $z = c+di \in \mathbf{C}$. Use the notation $Re(z) = c$ and $Im(z) = d$ to denote the real and imaginary part of z although I would caution that $Im(z) \in \mathbf{R}$. In addition this conflicts with our definition $Im(f)$ referring to the image of the function f. Sigh. We note that $z + \bar{z} \in \mathbf{R}$. Since we are going to get a real solution we can deduce that in the expression

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = a_1e^{it}\begin{bmatrix} i \\ -1 \end{bmatrix} + a_2e^{-it}\begin{bmatrix} -i \\ -1 \end{bmatrix}$$

that $\bar{a_1} = a_2$. We can get two different real solutions from the Real and Imaginary parts of one solution

$$e^{it}\begin{bmatrix} i \\ -1 \end{bmatrix} = (\cos t + i\sin t)\begin{bmatrix} i \\ -1 \end{bmatrix} = \begin{bmatrix} -\sin t \\ -\cos t \end{bmatrix} + i\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$$

$$Re(e^{it}\begin{bmatrix} i \\ -1 \end{bmatrix}) = Re((\cos t + i\sin t)\begin{bmatrix} i \\ -1 \end{bmatrix}) = \begin{bmatrix} -\sin t \\ -\cos t \end{bmatrix}$$

$$Im(e^{it}\begin{bmatrix} i \\ -1 \end{bmatrix}) = Im((\cos t + i\sin t)\begin{bmatrix} i \\ -1 \end{bmatrix}) = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$$

You may verify that the real part comes from the choice $a_1 = 1/2$, $a_2 = 1/2$ and the imaginary part comes from the choice $a_1 = -i/2$, $a_2 = i/2$. We now solve taking a linear combination of these two solutions (which are both real although their origin was complex):

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = a\begin{bmatrix} -\sin t \\ -\cos t \end{bmatrix} + b\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}, \quad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

We solve and get $a = 0$, $b = 1$ yielding the solution $y_1(t) = \cos t$, $y_2(t) = -\sin t$.

It is not particularly helpful to note that we can compute e^{At} for this A without using complex numbers. For this problem

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad A^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

from which we have expresssions for all powers of A. Then

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t \\ -t & 0 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} -t^2 & 0 \\ 0 & -t^2 \end{bmatrix} + \frac{1}{3!}\begin{bmatrix} 0 & -t^3 \\ t^3 & 0 \end{bmatrix} + \frac{1}{4!}\begin{bmatrix} t^4 & 0 \\ 0 & t^4 \end{bmatrix} + \frac{1}{5!}\begin{bmatrix} 0 & t^5 \\ -t^5 & 0 \end{bmatrix} + \cdots$$

$$= \left[\begin{array}{c|c} 1 + 0 - \frac{1}{2!}t^2 + 0 + \frac{1}{4!}t^4 + 0\cdots & 0 + t + 0 - \frac{1}{3!}t^3 + 0 + \frac{1}{5!}t^5\cdots \\ \hline 0 + t + 0 - \frac{1}{3!}t^3 + 0 + \frac{1}{5!}t^5\cdots & 1 + 0 - \frac{1}{2!}t^2 + 0 + \frac{1}{4!}t^4 + 0\cdots \end{array}\right]$$

$$= \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}$$