MATH 223: Vector Spaces.

One of the big new concepts in MATH 223 is that of a *vector space*. This concept, as well as the concept of *linear independence* (or *linear dependence*), is initially quite challenging for students. Part of the problem is that the language can be a bit confusing. We use *vectors* to mean two different things. First, we are familiar with a vector as a $t$-tuple, forming entries in $\mathbf{R}^t$. This is not the generic *vector* of vector spaces although it is an example of a vector. Second, a vector is an object in a vector space ( a vector space is a set of vectors) which supports operations of addition and scalar multiplication. The scalars must form a *Field* such as $\mathbf{R}$ (the real numbers), $\mathbf{C}$ (the complex numbers) and $\mathbf{Q}$ (the rational numbers). The vector addition and scalar multiplication must satisfy a list of axioms such as distributivity laws (e.g. $k(\mathbf{u}+\mathbf{v}) = k\mathbf{u}+k\mathbf{v}$, $(k+l)\mathbf{u} = k\mathbf{u}+l\mathbf{u}$).

Much as linear transformations are a more general framework than matrix multiplication, vector spaces and the vectors they contain generalize objects we know including the notion of $\mathbf{R}^n$. When I was first introduced to vector spaces, the instructor Harry Davis suggested we think of vectors as apples and oranges for which we have defined the appropriate operations. Thus $3.5\text{apple} - 2\text{orange}$ is a typical vector in that vector space.

When we come to $t$-tuples as vectors we must think of the underlying vector and not the particular coordinate system we are using. We have already seen an example of this with white and blue coordinates in $\mathbf{R}^2$ where the underlying vector has an existence apart from its representation. This leads to the following which is written in a provocative way:

$$\text{A vector } \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \text{ is } \textbf{not} \text{ a vector but merely a representation of a vector}$$

One person pointed out that even the vector given could be viewed as an 'apple' in a vector space so in that case I guess it could be a vector but it is not a vector in the way you usually think of the 3-tuple.

We have three main examples of vectors we consider. We have set of $t$-tuples in $\mathbf{R}^t$ with the usual vector addition and scalar multiplication. We have sets of functions with the usual addition and scalar multiplication of functions. We also have sets of $k \times k$ matrices $\mathbf{R}^{k\times k}$ under the usual matrix addition and scalar multiplication. These are not the only possibilities. Moreover, given the same set of objects we wish to consider as vectors, we may have alternate definitions of addition or scalar multiplication yielding alternate vector spaces. Note how multiplication of vectors is not being considered.

For $V$ to be a vector space, we must also have the *closure* properties that for $\mathbf{u}, \mathbf{v} \in V$, we have $\mathbf{u}+\mathbf{v} \in V$, and for $\mathbf{u} \in V$ and $k$ a scalar, we have $k\mathbf{u} \in V$. Note that $0\mathbf{u} = \mathbf{0}$ and so $\mathbf{0} \in V$. Roughly speaking, the operations of addition and scalar multiplication have to be well defined within the vector space.

We have also discussed a subspace $U$ of a vector space $V$, which is a set of vectors $U \subseteq V$ which is at the same time a vector space (we need only verify closure for $U$ since the other axioms are inherited from $V$).

Now the dimension of a vector space is the cardinality of a basis (we assume this is finite, we do not dwell on infinite dimensional vector spaces). We do **not** imagine that a set $U$ of $t$-tuples which is a vector space has dimension $t$ although it is true that dimension of $\mathbf{R}^t$ is $t$ since we can identify a basis of $\mathbf{R}^t$ as $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_t\}$. A set $U$ of $t$-tuples which is a vector space has some dimension which we can verify is at most $t$ but it could be less. (How should you show that the dimension is at most $t$? Assume you have $t+1$ linear independent vectors in $U$ and derive a contradiction). Thus *dimension* is being used as a piece of mathematical terminology for vector spaces in the context of bases and does not refer some English meaning of dimension. Maybe we would have been better to have a separate term but this is not standard.

The following are important theorems in identifying a vector space after we have identified the main examples of vector spaces such as $n$-dimensional Real space $\mathbf{R}^n$, $m \times n$ matrices $M^{mn}$, polynomials of degree at most $k$ in a variable $x$ $P_k(x)$, continuous functions on the interval $(0,1)$ $C(0,1)$

**Theorem**. *Let $T = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ be a finite set of vectors in a vector space $V$. Then*

$$span(T) = \{\sum_{i=1}^{p} a_i \mathbf{v}_i \,:\, a_i \in \mathbf{R} \text{ for } i = 1, 2, \ldots, p\}$$

*is a vector space, a subspace of $V$.*

**Proof**: We need to consider closure under addition and scalar multiplication. Let $\mathbf{x}, \mathbf{y} \in \text{span}(T)$. Say $\mathbf{x} = \sum_{i=1}^{p} x_i \mathbf{v}_i$ and $\mathbf{y} = \sum_{i=1}^{p} y_i \mathbf{v}_i$. Then $\mathbf{x} + \mathbf{y} = \sum_{i=1}^{p} x_i \mathbf{v}_i + \sum_{i=1}^{p} y_i \mathbf{v}_i = \sum_{i=1}^{p} x_i \mathbf{v}_i + y_i \mathbf{v}_i$, using the commutavity axiom for vector addition repeatedly, Then $\mathbf{x} + \mathbf{y} = \sum_{i=1}^{p} (x_i + y_i) \mathbf{v}_i$ using distributivity axiom for scalar multiplication. But then $\mathbf{x} + \mathbf{y} \in \text{span}(T)$. Similarly we obtain that $t \cdot \mathbf{x} \in \text{span}(T)$ for $t \in \mathbf{R}$.

**Theorem**. $\{\mathbf{x} : A\mathbf{x} = \mathbf{0}\}$ *is a vector space and in fact in view of our results in Gaussian elimination either there in only one solution $\mathbf{x} = \mathbf{0}$ or the set of solutions is the span of a set of vectors whose cardinality is the number of free variables.*

**Theorem**. *If we have a subset $U$ of a vector space $V$, then $U$ is a vector space, under the operations of vector addition and scalar multiplication of $V$, if and only if $U$ satisfies* closure*:*

$$\text{for any } \mathbf{x}, \mathbf{y} \in U \text{ and } c \in \mathbf{R}, \text{ then } \mathbf{x} + \mathbf{y} \in U \text{ and } c\mathbf{x} \in U.$$

For example the vector $\mathbf{0} \in V$ must also be in $U$ for $U$ to be a vector space.