9.1: Taylor polynomials

Ex) Find the 3rd order Taylor polynomial for $\ln(x)$ centered at $x=1$.

3rd order Taylor polynomial:

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

Recall:

$0! = 1$

$3! = 3\cdot 2\cdot 1$

$4! = 4\cdot 3\cdot 2\cdot 1$

$n! = n\cdot(n-1)\cdots 3\cdot 2\cdot 1$

In our case, $a=1$, $f(x)=\ln(x)$.

First, find $f(1)$, $f'(1)$, $f''(1)$, $f'''(1)$. Then plug into $P_3(x)$.

$f(x) = \ln(x)$ $\qquad f(1) = \ln(1) = 0$

$f'(x) = \frac{d}{dx}\ln(x) = \frac{1}{x}$ $\qquad f'(1) = \frac{1}{1} = 1$

$f''(x) = \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$ $\qquad f''(1) = \frac{-1}{1^2} = -1$

$f'''(x) = \frac{d}{dx}\left(-\frac{1}{x^2}\right) = -(-2)\frac{1}{x^3}$ $\qquad f'''(1) = 2$

$$\Rightarrow P_3(x) = 0 + 1(x-1) + \frac{(-1)}{2}\cdot(x-1)^2 + \frac{2}{3!}(x-1)^3$$

$$\boxed{P_3(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}}$$

Approximately, what is $\ln(2)$?

$$\ln(2) \approx P_3(2) = (2-1) - \frac{(2-1)^2}{2} + \frac{(2-1)^3}{3}$$

$$= 1 - \frac{1}{2} + \frac{1}{3}$$

$$= \frac{6}{6} - \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$$

Ex) Find 3rd order Taylor polynomial for $\ln(x)$ centered at $x=2$.

$$P_3(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3$$

$$= \ln(2) + \frac{1}{2}(x-2) + \frac{\frac{-1}{2^2}}{2!}(x-2)^2 + \frac{\frac{2}{2^3}}{3!}(x-2)^3$$

$$\boxed{P_3(x) = \ln(2) + \frac{x-2}{2} - \frac{(x-2)^2}{8} + \frac{(x-2)^3}{24}}$$

Note: different answer when centered at different x.

Ex) Find 2nd order Taylor polynomial for e^x centered at $x=0$.

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

when $a=0$,

$$P_2(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2}x^2$$

<u>step 1:</u> Find $f(0)$, $f'(0)$, $f''(0)$.

$f(x) = e^x \qquad f(0) = e^0 = 1$

$f'(x) = \frac{d}{dx}e^x = e^x \qquad f'(0) = 1$

$f''(x) = \frac{d}{dx}e^x = e^x \qquad f''(0) = 1$

<u>Step 2:</u> Plug into $P_2(x)$.

$$\boxed{P_2(x) = 1 + 1\cdot x + \frac{1}{2}x^2 = 1 + x + \frac{x^2}{2}}$$

Ex) Find the fourth order Taylor polynomial of $\cos(x)$ centered at $x=0$.

Step 0: Write the general form for 4th order Taylor polynomial centered at $x=0$.

$$P_4(x) = f(0) + f'(0)\cdot x + \frac{f''(0)}{2!}\cdot x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

Step 1: Find $f(0)$, $f'(0)$, $f''(0)$, $f'''(0)$, $f^{(4)}(0)$.

$f(x) = \cos(x)$ $\qquad f(0) = 1$

$f'(x) = -\sin(x)$ $\qquad f'(0) = 0$

$f''(x) = -\cos(x)$ $\qquad f''(0) = -1$

$f'''(x) = \sin(x)$ $\qquad f'''(0) = 0$

$f^{(4)}(x) = \cos(x)$ $\qquad f^{(4)}(0) = 1$

Step 2: Plug into $P_4(x)$

$$P_4(x) = 1 + 0\cdot x + \frac{-1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4$$

$$\boxed{P_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!}}$$

Ex) Find the 3rd order Taylor polynomial of $\sin(x)$ centered at $x=0$.

Step 0:

$$P_3(x) = f(0) + f'(0)\cdot x + \frac{f''(0)\,x^2}{2!} + \frac{f'''(0)\,x^3}{3!}$$

Step 1: Find $f(0)$, $f'(0)$, $f''(0)$, $f'''(0)$

$f(x) = \sin(x)$ $\quad f(0) = 0$

$f'(x) = \cos(x)$ $\quad f'(0) = 1$

$f''(x) = -\sin(x)$ $\quad f''(0) = 0$

$f'''(x) = -\cos(x)$ $\quad f'''(0) = -1$

Step 3: Plug in

$$P_3(x) = 0 + 1\cdot x + \frac{0}{2!}x^2 + (-1)\cdot\frac{x^3}{3!}$$

$$\boxed{P_3(x) = x - \frac{x^3}{6}}$$