Final 2013 – Find the interval(s) on which $f(x) = x + \frac{1}{x}$ is increasing. Find the local extrema.

step 1: Find $f'(x)$:

$$f'(x) = 1 + \frac{-1}{x^2} = \frac{x^2 - 1}{x^2} = \frac{(x+1)(x-1)}{x^2}$$

Step 2: Find crit pts

Is 0 a crit pt? No. $0 \notin \text{domain}(f)$

crit pts: -1, 1

Step 3: Tabulate:

	$x < -1$	$-1 < x < 1$	$x > 1$
sign(f')	$+$ $(f'(x) > 0)$	$-$	$+$

$\boxed{-1}$ $\boxed{1}$

Note: $\text{sign}(f') = \text{sign}\big((x+1)(x-1)\big) = \text{sign}(x+1) \cdot \text{sign}(x-1)$

$\Rightarrow f$ is increasing for $x < -1$ & $x > 1$

Local extrema: Local max at $x = -1$

Local min at $x = 1$

(by 1st derivative test)

Final 2011 – Consider the function $f(x) = \frac{x^3+x^2-2x-3}{x^2-3}$. It's first derivative is

$$f'(x) = \frac{(x^2-1)(x^2-6)}{(x^2-3)^2}.$$

On which intervals is f increasing/decreasing? Identify local maxima/minima. (just the x coordinate.)

Step 1: Find x values where f' changes sign.

$$\text{sign}(f'(x)) = \text{sign}\big((x^2-1)(x^2-6)\big)$$
$$= \text{sign}\big((x-1)(x+1)(x-\sqrt{6})(x+\sqrt{6})\big)$$

$\Rightarrow$ f' changes sign at $x = 1, -1, \sqrt{6}, -\sqrt{6}$

Step 2: Tabulate

	$x < -\sqrt{6}$	$-\sqrt{6} \le x \le -1$	$-1 \le x \le 1$	$1 \le x < \sqrt{6}$	$x > \sqrt{6}$
sign(f')	$+$	$-$	$+$	$-$	$+$

hint: $\text{sign}(f'(-10)) = (-)\cdot(-)\cdot(-)\cdot(-) = +$

increasing

decreasing

$\boxed{\begin{array}{l}\text{Local max's at: } -\sqrt{6},\ 1 \\ \text{Local min's at: } -1,\ \sqrt{6}\end{array}}$

Ex) Find absolute extremum of $f(x) = x^x$, on its domain which $(0, \infty)$.

Step 1: Find $f'(x)$. Use ln trick:

$$\ln f(x) = \ln(x^x) = x \ln(x)$$

$$\Rightarrow \frac{d}{dx} \ln f(x) = \frac{d}{dx}(x \ln(x))$$

$$\Rightarrow \frac{f'(x)}{f(x)} = \ln x + x \cdot \frac{1}{x} = \ln x + 1$$

$$\Rightarrow \boxed{f'(x) = (\ln x + 1) \cdot x^x}$$

Step 2: Crst pt. when $\ln x + 1 = 0$

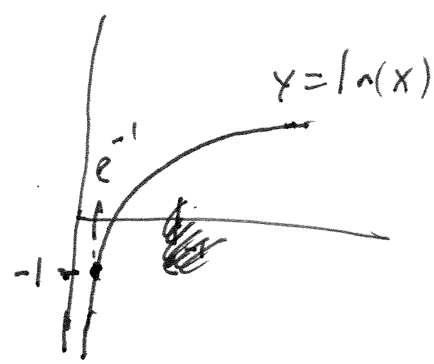
$$\Rightarrow \ln x = -1$$

$$\Rightarrow e^{\ln x} = e^{-1}$$

$$\Rightarrow x = e^{-1}$$

Step 3: Tabulate.

	$x < e^{-1}$	$x > e^{-1}$
sign(f')	$-$	$+$



↑ Local min at $x = e^{-1}$.

By thm, this is an absolute min.

Absolute extremum $\boxed{f(e^{-1}) = (e^{-1})^{(e^{-1})}}$

Final 2011 – Consider again the function $f(x) = \frac{x^3+x^2-2x-3}{x^2-3}$. It's first and second derivatives are given by

$$f'(x) = \frac{(x^2-1)(x^2-6)}{(x^2-3)^2}, \qquad f''(x) = \frac{2x(x^2+9)}{(x^2-3)^3}.$$

On which intervals is f(x) concave up? On which intervals is f(x) concave down? Find the coordinates of all local maxima, local minima, and inflection points.

For concavity, tabulate.

~~Step~~

$$\operatorname{sign}(f''(x)) = \operatorname{sign}(2x) \cdot \operatorname{sign}\left(\frac{1}{(x^2-3)^3}\right)$$

$$= \operatorname{sign}(2x) \cdot \operatorname{sign}(x^2-3)$$

$$= \operatorname{sign}(x) \cdot \operatorname{sign}(x-\sqrt{3}) \cdot \operatorname{sign}(x+\sqrt{3})$$

	$x < -\sqrt{3}$	$-\sqrt{3} < x < 0$	$0 < x < \sqrt{3}$	$x > \sqrt{3}$
sign(f'')	$-$	$+$	$-$	$+$

concave down concave up

concave down for $x \in (-\infty, -\sqrt{3})$ & $x \in (0, \sqrt{3})$

concave up for $x \in (-\sqrt{3}, 0)$ & $x \in (\sqrt{3}, \infty)$

Inflection points: (where f is defined, & f'' changes sign)

x values are just $x = 0$.

Coordinates of inflection pt.

$(0, f(0)) = (0, 1)$

Coordinates of local extrema:

Step 1. Crit pts where $f' = 0$.

(since f' is defined for all x in domain(f))

$$0 = f'(x) = \frac{(x^2-1)(x^2-6)}{(x^2-3)^2}$$

$\Rightarrow x = -1, 1, -\sqrt{6}, \sqrt{6}$ are crit pts.

Step 2

Use 2nd derivative test to determine if they are local max's or mins.

$\cancel{f''(-1) =}$ $\text{sign}(f''(-1)) > 0$ from table

$\Rightarrow -1$ is x coordinate of local min.

$\boxed{(-1, f(-1))}$ are coordinates of 1st local min.

$= (-1, \frac{1}{2})$

Similarly, $\text{sign}(f''(1)) < 0 \Rightarrow (1, f(1)) = (1, \frac{-3}{2})$ is a local max.

$\text{sign}(f''(\sqrt{6})) > 0 \Rightarrow (\sqrt{6}, f(\sqrt{6}))$ is a local ~~max~~ min

$\text{sign}(f''(-\sqrt{6})) < 0 \Rightarrow (-\sqrt{6}, f(-\sqrt{6}))$ is a local max