1. Let \( A = \begin{bmatrix} 0 & x \\ 1 & y \end{bmatrix} \) and \( B = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \). Determine all \( x, y \) so that \( AB = BA \).

2. Find a \( 2 \times 2 \) matrix \( A \), no entry of which is 0, with \( A^2 = A \). Note that your first guesses \( A = I \) or \( A = 0 \) (or indeed \( A = E_{11} \)) have 0 entries.

3. Darryl has a sum of \( s \) dollars in \( t \) coins. The coins are either \( $.25 \) or \( $.10 \) Express the number \( x \) of quarters (\$.25 coins) and the number \( y \) of dimes (\$.1 coins) in terms of \( s \) and \( t \).

4. Assume you are given a pair of matrices \( A, B \) which satisfy \( AB = BA \). Show that if we set \( C = A^2 + 2A \) and \( D = B^3 + 5I \), then \( CD = DC \). Then try to generalize this in some interesting way, namely find a property so that for matrices \( C, D \) with that certain property, then \( CD = DC \). For example \( C = A^2 + 6A \) and \( D = 3B^3 - 2I \) will also have \( CD = DC \).

5. Let \( R(\theta) \) denote the matrix of the transformation which rotates the plane by \( \theta \) counterclockwise around the origin. Explain in words (using transformations) why \( R(\theta)R(\phi) = R(\theta + \phi) \). Show how you can use this to derive the formulas for \( \cos(\theta + \phi), \sin(\theta + \phi) \) in terms of \( \cos(\theta), \sin(\theta), \cos(\phi), \sin(\phi) \).

6. Find the matrix \( A \) associated with the linear transformation \( T \) that has \( T \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \) and \( T \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} \).

7. a) Assume \( A, B \) are \( 2 \times 2 \) invertible matrices so that \( A^{-1} \) and \( B^{-1} \) exist. Show that \( (AB)^{-1} = B^{-1}A^{-1} \).

b) Given
\[
A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\]
then define \( A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \), where \( A^T \) is called the transpose of \( A \). The dot product of two vectors \( \mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}, \mathbf{y} = \begin{pmatrix} c \\ d \end{pmatrix} \) is \( \mathbf{x} \cdot \mathbf{y} = ac + bd \). Then the \( i, j \) entry of \( AB \) is the dot product of the \( i \)th row of \( A \) and the \( j \)th column of \( B \). Using this idea, show that \( (AB)^T = B^TA^T \). (One could verify \( (AB)^T = B^TA^T \) for two arbitrary \( 2 \times 2 \) matrices \( A, B \) directly but the argument wouldn’t generalize to larger matrices).

8. Consider two nonzero vectors \( \mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}, \mathbf{y} = \begin{pmatrix} c \\ d \end{pmatrix} \). Then there is a \( \theta \) with \( 0 \leq \theta < 2\pi \) and a \( \rho > 0 \) so that \( \mathbf{y} = \rho R(\theta)\mathbf{x} \). Use our knowledge of rotation matrices to establish a simple condition on \( a, b, c, d \) so that the angle \( \theta \) satisfies \( 0 < \theta < \pi \). You may assume \( a, b, c, d \) are nonzero, if that assists you, and even assume the two vectors \( \mathbf{x}, \mathbf{y} \) have the same length (\( \rho = 1 \)).